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THE EFFECT OF PERIODIC LONGITUDINAL NONUNIFORMITY ON THE FORMATION OF WAVEFRONTS FOR SELF PROPAGATING RADIATION IN GRADIENT WAVEGUIDES

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Abstract—A quasimode representation has been introduced with the aim of analysing self-propagating radiation in gradient waveguides periodically non-uniform in the longitudinal direction. The quasimodes have been demonstrated to form, in the paraxial approximation, a complete set of orthogonal basis functions at each point of the longitudinal axis. For radiation propagating in such waveguides effective controllability of the wavefront has been established.

Information processing by means of integrated optics is of great current practical interest, due to the latter's operational speed, capacity and miniaturization capabilities. The ability to form wave packets with a specified wavefront is rather important in this context. Provided the mode composition of the radiation passing down an optical waveguide is given, the transformation coefficients between various modes are normally introduced for a waveguide which is uniform in the longitudinal (i.e. propagation) direction [1]. The goal of the present work is to investigate, in the paraxial approximation, the effect of periodic longitudinal large-scale nonuniformities ($T \gg \lambda$, λ being the wavelength, T the period of the nonuniformity) on the form of wavefronts self-propagating in a gradient waveguide. In tackling this problem it is convenient to introduce a quasimodal representation for the nonuniform, longitudinal, periodic gradient-wave guide. By this we mean the solution of Maxwell's equations satisfying all the boundary conditions, namely,

$$\begin{aligned} E^z(x_1, x_2, z; t) &= E_0^{\bar{z}}(x_1, x_2, z) \exp\{i(\omega t - \tilde{\beta} z)\} \\ H^z(x_1, x_2, z; t) &= H_0^{\bar{z}}(x_1, x_2, z) \exp\{i(\omega t - \tilde{\beta} z)\}, \end{aligned} \tag{1}$$

wherein the eigenvalue $\tilde{\beta}$ is the propagation (quasi)-constant and $E_0^{\bar{z}}$, $H_0^{\bar{z}}$ are functions periodic in z, with periodicity T. By making the substitution $z \to z + T$ the quasimodes are seen to satisfy the condition

$$\begin{aligned} E(x_1, x_2, z + T) &= \bar{E}(x_1, x_2, z) \exp\{-i\tilde{\beta} T\} \\ \bar{H}(x_1, x_2, z + T) &= \bar{H}(x_1, x_2, z) \exp\{-i\beta T\}, \end{aligned} \tag{2}$$

i.e. there is phase repetition over T equal to the periodicity of the longitudinal nonuniformity, matching the mode-repetition distance $z = 2\pi N/\beta$ (β is the propagation constant). If the medium is weakly nonuniform $\lambda|\nabla n|/n \ll 1$, n = refractive index of the guide), Maxwell's equations reduce to the scalar Helmholtz equation [1]:

$$\frac{\partial^2 E}{\partial x_1^2} + \frac{\partial^2 E}{\partial x_2^2} + \frac{\partial^2 E}{\partial z^2} + k^2 n^2(x_1, x_2, z) E = 0, \tag{3}$$

where $\{x_1, x_2, z\}$ are Cartesian coordinates, E is one of the field components, and $k = 2\pi/\lambda_0$ is the wave number in vacuum.

Substituting the quasimode field intensity (1) into (3), we obtain the equation satisfied by the quasimodes:

$$\frac{\partial^2 E^z}{\partial x_1^2} + \frac{\partial^2 E^z}{\partial x_2^2} + \frac{\partial^2 E^z}{\partial z^2} - 2i\tilde{\beta} \frac{\partial E^2}{\partial z} + [k^2 n^2(x_1, x_2, z) - \tilde{\beta}^2] E^z = 0. \tag{4}$$

The set of propagating quasimodes are determined by the boundary conditions. If there is longitudinal nonuniformity the quasimodes reduce to the ordinary guide modes, with a similar reduction for the corresponding dispersion relations. From a mathematical point of view the quasimodes of the present aperture are analogous, in the paraxial approximation, to the quasienergetic states (QES) of quantum systems describable by Schroedinger equations with time-periodic Hamiltonians [2, 3]. This analogy is easily established by considering that in the paraxial approximation the Helmholtz equation (3) may be transformed into a nonstationary Schroedinger equation, where the longitudinal coordinate ξ replaces the time, and $h \leftrightarrow 1/k$ [4]:

$$\frac{\mathrm{i}}{k}\frac{\partial\psi}{\partial\xi} = \frac{1}{2k^2}\left(\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2}\right) + \frac{1}{2}(n_0^2 - n^2)\psi \equiv H\psi, \tag{5}$$

where

$$\xi = \int_0^z n_0^{-1}(z)\,\mathrm{d}z; \qquad n_0 = n(0, 0, z)$$

$$\psi(x_1, x_2, z) = n_0^{1/2}E(x_1, x_2, z)\exp\left\{-ik\int_0^z n_0(z)\,\mathrm{d}z\right\}; \tag{6}$$

$\xi = \int_0^z n_0^{-1}(z)\,\mathrm{d}z$; and $n_0 = n(0, 0, z)$ is the refractive index along the z-axis. Hence, in view of (1) and (6), in the paraxial approximation, quasimodes like the QES form a complete orthogonal set at each point along the z-axis. This allows one to expand an arbitrary field in such a waveguide in terms of a complete set of quasimodes at every point on the z-axis, and to reduce the problem to investigating the quasimode properties. Because of their properties one may consider quasimodes as modes in effect of a longitudinal uniform waveguide. Equation (5) may be solved by the usual quantum mechanical methods, namely in terms of integrals of motion, coherent states, or by the dynamical symmetry group method [5, 6]. As a concrete example let us consider a planar graded index guide with a parabolic profile,

$$n^2(x, \xi) = n_0^2(x, \xi) - \omega^2(\xi)x^2 - 2f(\xi)x, \tag{7}$$

where $\omega^2(\xi)$ is the gradient parameter and $f(\xi)$, a function describing the bending of the guide axis, is a periodic function of ξ with period $T_\xi = T'$. An explicit expression for the quasimode field intensity may be obtained by solving the equation of motion for the classical oscillator:

$$\ddot{\varepsilon} + \omega^2(\xi)\varepsilon = 0; \qquad \omega^2(\xi + T') = \omega^2(\xi). \tag{8}$$

By Floquet's theorem the solutions of Eq. (8) may be either stable or unstable. Stable solutions have the form

$$\varepsilon(\xi + T') = \varepsilon(\xi)\exp\{\mathrm{i}\chi T'k\}; \qquad \varepsilon^*(\xi + T') = \varepsilon^*(\xi)\exp\{-\mathrm{i}\chi T'k\}, \tag{9}$$

where the star represents differentiation with respect to ξ; and $\chi = \frac{1}{T'k}\cdot\int_0^{T'}|\varepsilon|^{-2}\,\mathrm{d}\tau$ is a real number.

Following [5, 7] we derive the following expression for the field intensity of a directed quasimode within the parabolically graded longitudinally nonuniform waveguide:

$$E_n(z) = n_0^{-1/2}\left(\frac{\varepsilon^*}{2\varepsilon}\right)^{n/2}(n!\pi^{1/2}k^{-1/2}\varepsilon)\exp\left\{ik\int_0^z n_0(z)\,\mathrm{d}z\right\}\exp\left\{\frac{\mathrm{i}\varepsilon'}{2\varepsilon}kn_0(x-\eta)^2\right\}$$
$$\times\exp\left\{\mathrm{i}\eta' n_0 k(x-\eta) + \mathrm{i}k\int_0^z\left[\frac{1}{2}\eta'^2 n_0^2 - \frac{1}{2}\omega^2\eta^2 + f(\tau)\eta\right]\frac{\mathrm{d}\tau}{n_0(\tau)}\right\}H_n\left(\frac{\sqrt{k}(x-\eta)}{|\varepsilon|}\right), \tag{10}$$

where H_n is the Hermite polynomial, and η is the real solution of the equation

$$\ddot{\eta} + \omega^2(\xi)\eta = f(\xi). \tag{11}$$

By choosing a periodic solution of (11) we easily find the dispersion relations for the propagation (quasi) constants $\tilde{\beta}$:

$$\tilde{\beta}_n = k\langle n_0\rangle - k\langle 1/n_0\rangle\chi(n + 1/2) - \Delta\tilde{\beta}, \tag{12}$$

where

$$\Delta\tilde{\beta}=\frac{k\langle 1/n_0\rangle}{T'}\cdot\int_0^{T'}\left[\frac{1}{2}\dot{\eta}^2-\frac{1}{2}\omega^2(\xi)\eta^2+f(\varepsilon)\eta\right]d\xi,$$
$$\langle n_0\rangle=\frac{1}{T}\cdot\int_0^{T'}n_0(z)\,dz;\qquad \langle 1/n_0\rangle=\frac{1}{T}\int_0^{T}n_0^{-1}(z)\,dz. \tag{13}$$

The dispersion spectrum $\tilde{\beta}_n$ of (12) turns out to be enti discrete for guided quasimodes. Radiation quasimodes corresponding to unstable solutions of (8),

$$\varepsilon_1(\xi+T')=\varepsilon_1(\xi)\exp\{-\chi T'k\};\qquad \varepsilon_2(\xi+T')=\varepsilon_2(\xi)\exp\{\chi T'k\}, \tag{14}$$

where ε_1, ε_2 are two linearly independent solutions of (8) satisfying $\varepsilon_2\dot{\varepsilon}_1-\varepsilon_1\dot{\varepsilon}_2=1$, $\chi^2>0$, have the following form:

$$E_{v1}\pm(x,z)=n_0^{-1/2}|0,z\rangle\Gamma\left(\frac{1}{2}-\mathrm{i}v\right)\exp\left\{\frac{\mathrm{i}k(x-\eta)^2}{4\varepsilon_1\varepsilon_2}\right\}\left(\mathrm{i}k\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1/4+\mathrm{i}v}\exp\left\{\mathrm{i}k\int_0^z n_0\,dz\right\}$$
$$\times\exp\left\{\mathrm{i}\eta' n_0 k(x-\eta)+\mathrm{i}k\int_0^z\left[\frac{1}{2}\eta'^2 n_0-\frac{1}{2}\omega^2\eta^2+f(\tau)\eta\right]\frac{d\tau}{n_0}\tau\right\}D_{-1/2+\mathrm{i}v}\left(\frac{\pm\sqrt{\mathrm{i}k(x-\eta)}}{\sqrt{\varepsilon_1\varepsilon_2}}\right), \tag{15}$$

where $D_{-1/2+\mathrm{i}v}$ is the parabolic cylinder function, v is a real number and the vacuum state $|0,z\rangle=\left(k/2\pi\varepsilon_1^{1/2}\exp\left\{\frac{\mathrm{i}k\varepsilon_1'}{2\varepsilon_1}n_0x^2\right\}\right)$ (the prime on ε, η denotes differentiation with respect to z).

For radiation quasimodes the dispersion spectrum changes to

$$\tilde{\beta}=k\langle n_0\rangle-k\langle 1/n_0\rangle\chi v-\Delta\tilde{\beta}, \tag{16}$$

where $\Delta\tilde{\beta}$, as per (13), is continuous and doubly degenerate.

The continuous nature of (16) is due to the fact that, by (15), in the unstable region described by (14) the medium through which radiation is propagating will assume waveguide properties, and will radiate the light incident on it in all directions over the $[x,z]$-plane. It follows from (12), (13) and (16) that the nature of the solutions is unaltered by periodic bending of the waveguide axis, which only causes a shift $\Delta\tilde{\beta}$ in the quantity $\tilde{\beta}$. The shift $\Delta\tilde{\beta}$ with respect to the propagation constants β_n may be estimated by using (12). Thus, if the gradient parameter $\omega(\xi)$ is a function which differs little from the constant ω_0 representing the gradient parameter of a longitudinally uniform wavelength, i.e. if

$$\omega^2(\xi)=\omega_0^2(1+4h\cos\omega\xi), \tag{17}$$

where $\omega=2\pi/T'$, $|4h|\ll 1$, then as $h\to 0$ the size of the quantum χ is given by [7]:

$$\chi=\frac{\omega_0}{k}\cdot\begin{cases}\sqrt{\varepsilon^2-h^2}, & \omega=2\omega_0(1+\varepsilon),\qquad \varepsilon\to 0\\ 1-\dfrac{4\omega_0^2h^2}{4\omega_0^2-\omega^2}. & \end{cases} \tag{18}$$

If ω is restricted to be nearly $2\omega_0$, i.e. if only first-order terms of h and ε are retained, the range of variation of ω associated with the stability region may be determined:

$$\omega_0<\omega<2\omega_0(1-|h|);\qquad -\frac{1}{2}<\varepsilon<-|h|;\qquad h=\frac{\delta\omega}{2\omega_0}=\frac{\delta n\cdot n_0}{2\omega_0^2a^2}, \tag{19}$$

where δn_0 is the change induced in the refractive index n_0 along the waveguide axis, $\delta\omega$ is the amplitude of gradient parameter change, $\omega(\xi)=\omega_0+\delta\omega\cos\omega\xi$, and a is the waveguide depth.

Hence the spectrum shift,

$$\Delta\beta_m=\beta_m-\tilde{\beta}_m=\frac{\omega_0}{n_0}\left(m+\frac{1}{2}\right)[\sqrt{\varepsilon^2-h^2}-1] \tag{20}$$

is proportional to the quasimode number m (n_0 is assumed to be independent of ξ).

If ω is far from $2\omega_0$ the spectral shift is determined by the following:

$$\Delta\beta_m = \beta_m - \tilde{\beta}_m = -\frac{\omega_0}{n_0}\left(m + \frac{1}{2}\right)\cdot\frac{4\omega_0^2 h^2}{4\omega_0^2 - \omega^2}. \tag{21}$$

The shifts (20) and (21) have opposite signs. $\Delta\beta$ depends essentially on the parameter h, which is essentially determined by the induced refractive index change δn_0 along the waveguide axis and is limited by technical considerations. For example, in a waveguide formed in $LiNbO_3$ the index on the axis is $n_0 = \Delta n + n_s \approx 2.2$, where $\Delta n = 2 \times 10^{-2}$ is the index gradient and $n_s = n_e = 2.17$ is the refractive index on the optic axis of $LiNbO_3$ having a depth of $a = 5\ \mu$m, $\delta n_0 \sim 10^{-5}$, the gradient parameter $\omega_0 \approx \sqrt{2n_0\Delta n}/a = 8 \times 10^{-2}$, so that $h \sim 10^{-4}$. An increase of $h(\delta n_0)$ at constant ε means that the lowest modes are more easily excited. We note that in the limit that $\varepsilon = h$, χ vanishes, which corresponds to the boundary between the stable and unstable regions of (8). In that case the guided quasimodes become radiative, the spectrum remaining continuous. We may estimate the angular spectral shift by using the prismatic method for the input/output radiation of the waveguide:

$$\Delta\Theta_m = \Theta_m - \tilde{\Theta}_m = \arcsin\frac{\beta_m}{kn_{pr}} - \arcsin\frac{\tilde{\beta}_m}{kn_{pr}}. \tag{22}$$

Hence, when ω is near $2\omega_0$ and as $\varepsilon \to h$, $\Delta\Theta_m$ for the zeroth quasimode becomes $\Delta\Theta_0 = 6'$ of arc, while for $m = 5$, $\Delta\Theta_5 = 1°$, so that the angular separation of neighbouring modes is $\Delta\Theta_{0,1} = 10'$ (the refractive index of the prism $n_{pr} \approx 2.5$; $\lambda = 0.63\ \mu$m and $\Delta\Theta_{0,1} = \Theta_0 - \Theta_1$). Therefore, by changing the periodicity of the graded-guide nonuniformity, one can control the spectrum of propagation (quasi) constants, exciting thereby various groups of modes corresponding to an effective longitudinally uniform waveguide, and allowing for the formation of various wavefronts propagating down the graded guide. Note that any deviation from a parabolic index profile will upset the equidistant spacing of the propagation (quasi) constants [12] and will generate an inter-quasi mode dispersion $d\tilde{\beta}/dk$, where k is the wave vector. Therefore, the form of the signal transmitted by the optical waveguide appears to be controllable, a facility of practical interest in fibre optics communication.

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