Inverse scattering transform algorithm for the Manakov system

A.E. Chernyavsky [1], *L.L. Frumin* [1]
[1] *Institute of Automation and Electrometry SB RAS,*
630090, Novosibirsk, Russia, Academician Koptyug Ave., 1

Abstract

A numerical algorithm is described for solving the inverse spectral scattering problem associated with the Manakov model of the vector nonlinear Schrödinger equation. This model of wave processes simultaneously considers dispersion, nonlinearity and polarization effects. It is in demand in nonlinear physical optics and is especially perspective for describing optical radiation propagation through the fiber communication lines. In the presented algorithm, the solution to the inverse scattering problem based on the inversion of a set of nested matrices of the discretized system of Gelfand–Levitan–Marchenko integral equations, using a block version of the Levinson-type Toeplitz bordering algorithm. Numerical tests carried out by comparing calculations with known exact analytical solutions confirm the stability and second order of accuracy of the proposed algorithm. We also give an example of the algorithm application to simulate the collision of a differently polarized pair of Manakov optical vector solitons.

Keywords: Manakov system, polarization, dispersion, inverse scattering problem, algorithm.

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Introduction

Investigations of the polarization effects complicated by wave dispersion and nonlinearity are of great importance for modern optical technologies and nonlinear optics. Manakov [1], studying the phenomena of self-focusing and self-induced transparency of polarized light beams in nonlinear and dispersive optical media, described the vector version of the nonlinear Schrödinger equation, known as the Manakov model. The model comprises a pair of nonlinear Schrödinger equations with second-order dispersion and cubic (Kerr) nonlinearity for two optical polarizations. This model of wave processes simultaneously considers dispersion, nonlinearity and polarization effects, is in demand in nonlinear physical optics, and especially for describing optical radiation propagation through the fiber optical communication lines (FOCL) [2].

Modern technologies for receiving and transmitting information over the FOCL use modulation of low-frequency envelope of the electric field of light wave. Low (compared to optical) frequency modulation of the envelope allows the use of electronics in the processes of coding/decoding information. The electric field envelope of optical radiation in standard single mode fiber (SMF) permits amplitude and phase modulation, and also enables to use two polarizations to increase the FOCL performance. Increased fibre channels performance leads to growth of envelope frequency and amplitude, making it necessary to account for dispersion and nonlinear effects that distort information. Nonlinear and dispersion distortions in the scalar approximation are described by the nonlinear Schrödinger equation for the field envelope [2]. The dispersion effects in the fiber lead to envelope wave packet blurring, and nonlinear effects cause nonlinear phase overruns, and other effects discussed in [2]. The interaction of dispersion and nonlinearity generates solitons, where the dispersion broadening is compensated by the nonlinear compression of the wave front. Polarization of radiation even more complicates the problem and creates new distorting information effects, in particular, polarization mode dispersion (PMD), when the two envelope polarizations have different velocities.

The applicability of the Manakov model for the common description of nonlinear and dispersion effects, and polarization effects in fiber was studied in the works of Menyuk (See [3] and references there), where it was shown that excluding the attenuation/gain and neglecting PMD, the electric field envelope $\boldsymbol{E}$ with polarization components $\{E_1, E_2\}$ is described by the following vector equation, of Manakov type:

$$\mathrm{i}\frac{\partial \boldsymbol{E}}{\partial Z}+\beta_2\frac{1}{2}\frac{\partial^2 \boldsymbol{E}}{\partial T^2}-\gamma\frac{8}{9}|\boldsymbol{E}|^2\,\boldsymbol{E}=0, \qquad (1)$$

with Z being a distance along the fiber, and T is time in the frame moving together with the the field envelope. The parameter β_2 is the characteristic of dispersion. Note that it is negative for the anomalous dispersion and positive for the normal dispersion in the fiber. In the main focusing range β_2 is usually assume equal $-22\,ps^2/km$. The parameter γ is the nonlinear Kerr coefficient. Typical value of γ in fiber is $1.3W^{-1}\,km^{-1}$. Follow [4] and choosing the characteristic line length Z_0 and using the values of time $T_0=\sqrt{(|\beta_2|Z_0)/2}$ and field $E_0=\sqrt{2/\left(\frac{8}{9}\gamma Z_0\right)}$ as units, we define dimensionless parameters $t=T/T_0$ and

$\boldsymbol{q}=\boldsymbol{E}/\boldsymbol{E}_0$, for which the equation (1) becomes the dimensionless form of Manakov equation:

$$\mathrm{i}\frac{\partial \boldsymbol{q}}{\partial z}+\frac{\partial^2 \boldsymbol{q}}{\partial t^2}-2\sigma\,|\,\boldsymbol{q}\,|^2\,\boldsymbol{q}=0, \tag{2}$$

where the vector $\boldsymbol{q}=\{q_1(z,t),q_2(z,t)\}$ describes two normalized polarization components of the optical wave envelope. The constant σ in (2) is equal to +1 or –1 and corresponds to normal (defocusing) and anomalous (focusing) fiber dispersion, respectively. This Manakov equation as a vector form of the nonlinear Schrödinger equation, belongs to a nontrivial class of integrable nonlinear partial differential equations investigated by the inverse scattering transform method (ISTM) [5, 6]. The essence of the ISTM, an outstanding achievement of modern mathematical physics, is that the solution of a nonlinear equation is reduced to the study of linear operator spectral problems (direct and inverse), connected (associated) with the original nonlinear equation. The ISTM is also known as the Nonlinear Fourier Transform (NFT) because of the similarity to the conventional Fourier transform.

For applying the ISTM, Manakov constructed the following linear system of equations (the Manakov system) for the spectral problems:

$$\frac{\partial f_1}{\partial t}+\mathrm{i}\lambda f_1=q_1 f_2+q_2 f_3,$$
$$\frac{\partial f_2}{\partial t}-\mathrm{i}\lambda f_2=\sigma q_1^* f_1,\ \frac{\partial f_3}{\partial t}-\mathrm{i}\lambda f_3=\sigma q_2^* f_1. \tag{3}$$

Here λ is the complex eigenvalue number, (f_1,f_2,f_3) is the 3-dimensional eigenvector, and the asterisk means complex conjugate. In the defocusing case, the spectrum of the Manakov system contains only the continuous spectrum corresponding to dispersive waves. In the focusing case ($\sigma=-1$), a discrete spectrum can be added to the continuous spectrum. This discrete spectrum corresponds to the soliton solutions.

In the frame of the ISTM the solutions of the relatively simple Manakov system (3) are also the solutions of the nonlinear vector Schrödinger equation (2) of the Manakov model for an arbitrary fixed z. The evolution of the solution along the coordinate z is given by a simple phase factor [1, 5, 6]. We also note that the Manakov system turns into the Zakharov–Shabat system if one polarization absents and the vector Schrödinger equation becomes scalar. The Manakov system can be exploited to describe several other nonlinear problems, including the propagation of ultrashort polarized optical pulses in resonant two-level media [7].

The study of the scattering problems allows not only analytical investigations of the integrable nonlinear problems. The numerical implementation of the ISTM also represents an effective approach to solve the Cauchy problem of the nonlinear Schrödinger equation, with no iterations. ISTM for nonlinear equations with localized (decreasing at infinity) solutions leads to a system of linear integral Gelfand–Levitan–Marchenko equations (GLME), equivalent to the Manakov system.

For the case of more simple scalar Schrödinger equation, without polarization effects, associated with the well-known Zakharov–Shabat (ZS) system in our laboratory were developed efficient computational algorithms - Toeplitz Inner Bordering (TIB) [9, 10]. These algorithms are based on the direct numerical solution of the GLME and belong to modifications of the well-known Levinson's algorithm [11]. Their numerical efficiency is caused by the use of the Toeplitz symmetry of the discretized GLME system. The TIB algorithms were successfully applied to calculate the fiber Bragg gratings [12, 13], and solutions of inverse problems for the Helmholtz equation [14]. Recently in fiber-optic networks, these algorithms have found application to compensate for nonlinear-dispersive distortions of information signals and, in particular, for the development of new approaches in information transmission based on the integrability of nonlinear Schrödinger equation [15, 16, 17].

Due to the complexity of the problem, there are comparatively few works devoted directly to algorithms for the inverse scattering problem of the Manakov system: [18] proposes an iterative algorithm for solving GLME, but of the 1st order of accuracy; in [19] also the only first order of accuracy was achieved; [20] proposes a new perspective but not yet well known 2nd order algorithm, based on the Toeplitz decomposition of the block matrix of GLME system. The ISTM-integrable vector Schrödinger equation od the Manakov model is, of course, the basic model and it does not account for signal attenuation and polarization mode dispersion in fiber. However, this approach is in demand for numerical simulation of the information transfer in fiber-optical lines. We do not touch here on simpler direct scattering problems. The extension of the TIB algorithm to the inverse problems for the Manakov system is the main goal of the work.

1. GLME

The condition of rapidly decreasing at infinity solutions, leads for the Manakov system to a system of nine Gelfand-Levitan-Marchenko integral equations for the three three-components vectors $\boldsymbol{A}^{(0)},\boldsymbol{A}^{(1)},\boldsymbol{A}^{(2)}$ [7, 8, 19]. However, it turns out that we need only three equations for the first components of the vector functions $A_1^{(0)},A_1^{(1)},A_1^{(2)}$ for the inverse scattering problem (ISP) solution.

In the numerical algorithm, the ISP, under the condition of rapidly decreasing solutions, is on a finite interval, for example: $0\le t\le T$.

It is assumed that the GLME kernels $\Omega_1(t)$, $\Omega_2(t)$ vanish outside this interval, and the GLME for the left ISP take the following form:

$$A_1^{(0)*}(t,\tau)+\sum_{\alpha=1,2}\int_{-\tau}^{t}\Omega_\alpha(\tau+s)A_1^{(\alpha)}(t,s)ds=0, \tag{4}$$

$$\sigma A_1^{(\alpha)*}(t,\tau)+\int_{-\tau}^{t}\Omega_\alpha(\tau+s)A_1^{(0)}(t,s)ds=-\Omega_\alpha(t+\tau), \quad (5)$$

where $-t<\tau$, $s<t\leq T$, and $\alpha=1,2$.

The inverse scattering problem's solution $q_1(t)$, $q_2(t)$ for two orthogonal polarizations connected with solutions of GLME equations by the synthesizing relations:

$$q_\alpha(t)=2\sigma A_1^{(\alpha)*}(t,t-0),\ \alpha=1,2.$$

The first step in the GLME numerical solution is a replacement of variables which gives us the integral equations with difference arguments of the kernels and leads to matrix blocks with Toeplitz symmetry after the discretizathion of these equations. Following [9, 10] we carry out the complex conjugation of equation (4) and replace of unknown functions:

$$u(t,s)=A_1^{(0)}(t,t-s),\ v_\alpha(t,\tau)=\sigma A_1^{*(\alpha)}(t,\tau-t).$$

Here and hereinafter $\alpha=1,2$. Given these notations, we rewrite (4) in the form:

$$u(t,s)+\sigma\int_{s}^{2t}\sum_{\alpha=1,2}\Omega_\alpha^*(\tau-s)v_\alpha(t,\tau)d\tau=0, \quad (6)$$

$$v_\alpha(t,\tau)+\int_{0}^{\tau}\Omega_\alpha(\tau-s)u(t,s)ds=-\Omega_\alpha(\tau). \quad (7)$$

Also, we respectively rewrite synthesizing relations as $q_\alpha(t)=v_\alpha(t,2t-0)$.

2. Discretization of the GLME

Let us introduce a discrete computational grid: $t_m=hm/2$; $m=1,\ldots,N$; $h=2T/N$; $s_k=hk$; $\tau_n=hn$; $n,k=0,\ldots,m$.

Here h is the step of the grid. For the beginning we discretize equations (6), (7) with the 1st order of approximation accuracy. We replace the integrals in the equation (6) by the right Riemann sum:

$$\sum_{n=k+1}^{m}h\Omega_\alpha^*(\tau_n-s_k)v_\alpha(t_m,\tau_n)=\sum_{n=k+1}^{m}\mathrm{v}_{\alpha;n}^{(m)}h\Omega_{\alpha;n-k}^*.$$

Here roman letters denote the grid vectors: $\mathrm{v}_{\alpha;n}^{(m)}=v_\alpha(x_m,\tau_n)$ and $\Omega_{\alpha;n-k}^*=\Omega_\alpha^*(\tau_n-s_k)$. Since $n\geq k$, this sum is the multiplication of the upper triangular Toeplitz matrix Q_α with elements $Q_{\alpha;k,n}=h\Omega_{\alpha;n-k}^*$, with size $m\times m$, by the vector $\mathrm{v}_{\alpha;n}^{(m)}$. The discrete analog of the equation (6) now can be represented as:

$$\mathrm{u}_k^{(m)}+\sigma\sum_{n=k}^{m-1}\sum_{\alpha=1,2}Q_{\alpha;k,n}\mathrm{v}_{\alpha;n}^{(m)}=0, \quad (8)$$

where $k=1,\ldots,m$. The integral in equations (6) also can be represented using the right Riemann sum as products of low triangular Toeplitz matrix R_α with size $m\times m$ and with elements $R_{\alpha;n,k}=h\Omega_{\alpha;n-k}$, where $n\geq k$, on the vector $\mathrm{u}_k^{(m)}=u(x_m,\sigma_k)$. Note here that matrices Q_α are Hermitian conjugations (†) of matrices R_α: $Q_\alpha=R_\alpha^\dagger$. We write the discrete analog of equations (7) in the next form:

$$\mathrm{v}_{\alpha;n}^{(m)}+\sum_{k=1}^{n}R_{\alpha;n,k}\mathrm{u}_k^{(m)}=r_{\alpha,n}, \quad (9)$$

where the right part is $r_{\alpha,n}=-\Omega_{\alpha,n}$, and $n=1,\ldots,m$. With m changing from 1 to N we get N systems of linear equations (8),(9) with size of $3m\times 3m$.

3. First order accuracy Block TIB algorithm

With m changing from 1 to N equations (8), (9) describe N systems of linear equations. These systems are nested one into another that resembles a bordering numerical algorithm. For the numerical solution of the ISP, it is necessary to solve all the obtained nested systems and determine the potential vector components: $q_{\alpha;m}=2\mathrm{v}_{\alpha;m}^{(m)}$, $m=1,2,\ldots N$. Gaussian elimination method for N nested GLME systems requires $O(N^4)$ flops. For our opinion the best variant of the algorithm for solving such a series of nested linear systems seems to be Levinson-type bordering algorithm [11] which in the process of this bordering addresses all the systems and requires only $O(N^2)$ flops.

The complete system (8), (9) contains a matrix comprising nine Toeplitz blocks. Despite all these blocks being Toeplitz, the system's complete matrix, unlike the case of the ZS system, does not have Toeplitz symmetry.

We describe here an approach referred as a block one, based on the block notation of the discretized GLME. For the first order approximation accuracy, the total matrix G of the GLME can be represented as a block Toeplitz matrix comprising blocks - matrices of size 3×3. We note that this block matrix is Toeplitz only for the case of the first order accuracy approximation of the GLME. We write this system of equations in the block form:

$$\mathrm{G}\,\mathrm{w}^{(m)}=\mathrm{r}^{(m)}, \quad (10)$$

where

$$\mathrm{w}^{(m)}=\left(w_1^{(m)},w_2^{(m)},\ldots,w_m^{(m)}\right)^T,\ w_k^{(m)}=(u_k^{(m)},v_{1,k}^{(m)},v_{2,k}^{(m)})^T,$$

$$\mathrm{r}^{(m)}=\left(r_1^{(m)},\ldots,r_m^{(m)}\right)^T,\ r_k^{(m)}=(0,-\Omega_{1,k},-\Omega_{2,k})^T,$$

$$k=1,\ldots,m.$$

The symbol T here stands for transposition. Total block matrix G consists of blocks Ψ_n, $n=0,1,\ldots,m-1$, and its Hermitian conjugates, with size 3×3:

$$\mathrm{G}=\begin{pmatrix}\Psi_0 & \sigma\Psi_1^\dagger & \ldots & \ldots & \sigma\Psi_{m-1}^\dagger\\ \Psi_1 & \Psi_0 & \sigma\Psi_1^\dagger & \ldots & \sigma\Psi_{m-2}^\dagger\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \Psi_{m-1} & \Psi_{m-2} & \ldots & \Psi_1 & \Psi_0\end{pmatrix},\ m=1,..,N.$$

Here

$$\Psi_0=\begin{pmatrix}1 & \sigma h\Omega_{1;0}^* & \sigma h\Omega_{2;0}^*\\ h\Omega_{1;0} & 1 & 0\\ h\Omega_{2;0} & 0 & 1\end{pmatrix},\ \Psi_k=\begin{pmatrix}0 & 0 & 0\\ h\Omega_{1;k} & 0 & 0\\ h\Omega_{2;k} & 0 & 0\end{pmatrix},$$

where $k=1,\ldots,m-1$. This system of equations allows us to find the solution of the Manakov system with the first order of approximation accuracy.

We will omit the superscript $^{(m)}$ for almost all matrices and leave only for the first $g_1^{(m)}$ and last $g_m^{(m)}$ rows of the inverse matrix, that is necessary to describe the algorithm.

For a block Toeplitz matrix, there are well-known block algorithms for its inversion based on the bordering method of the Levinson algorithm type. See, for example [21]. For the matrix inversion, it is enough to find recurrently the upper $g_1^{(m)}$ and lower $g_m^{(m)}$ block rows of the inverse block Toeplitz matrix. These rows are the main auxiliary (block) arrays of the Levinson type inverse bordering block algorithm. Denote the elements of these block rows at the m-step of the block algorithm in the following form:

$$g_1^{(m)}=(X_0\ \ldots\ X_{m-1}),\ g_m^{(m)}=(Y_{m-1}\ \ldots\ Y_0). \tag{11}$$

Let us introduce the next set of 3×3 matrices for a more convenient notation of the algorithm:

$$P=(p_{i,j}),\, F=(f_{i,j}),\, \hat{P}=(\hat{p}_{i,j}),\, \hat{F}=(\hat{f}_{i,j}),\, i,j=1,2,3.$$

These matrices are determined by the sums:

$$\begin{aligned}
&F=\left(X_0\Psi_1+X_1\Psi_2+\ldots+X_{m-1}\Psi_m,\right),\\
&\hat{F}=\sigma\left(Y_{m-1}\Psi_m^\dagger+Y_{m-2}\Psi_{m-1}^\dagger+\ldots+Y_0\Psi_1^\dagger\right),\\
&P=\sigma\left(X_0\Psi_m^\dagger+X_1\Psi_{m-1}^\dagger+\ldots+X_{m-1}\Psi_1^\dagger,\right),\\
&\hat{P}=\left(Y_{m-1}\Psi_1+Y_{m-2}\Psi_2+\ldots+Y_0\Psi_m\right).
\end{aligned} \tag{12}$$

The sums make up the main loop of the algorithm, which is executed for each m from 1 to N. Note here that not all 36 elements of the matrices $P,F,\hat{P},\hat{F}$, but only some of them are used in the block algorithm.

The main step of the bordering algorithm is to calculate the first $g_1^{(m+1)}$ and last $g_{m+1}^{(m+1)}$ block rows of the inverse matrix based on their values $g_1^{(m)}$, $g_m^{(m)}$ on the previous step. Algorithms like Levinson's find new rows as a linear combination of the zero-padded rows of the previous step:

$$\begin{aligned}
&g_1^{(m+1)}=c^{(m)}\left(g_1^{(m)},0\right)+d^{(m)}\left(0,g_m^{(m)}\right),\\
&g_{m+1}^{(m+1)}=\hat{d}^{(m)}\left(g_1^{(m)},0\right)+\hat{c}^{(m)}\left(0,g_m^{(m)}\right),
\end{aligned} \tag{13}$$

where block 0 is zero 3×3 matrix. The coefficients $c^{(m)},\hat{c}^{(m)},d^{(m)},\hat{d}^{(m)}$ are determined uses the main properties of the first and last block rows of the inverse matrix. Namely, multiplying the first row by the system matrix G results in the first row, and multiplying the last row, respectively, gives the last row of the block identity matrix. The result is the following system of equations for the block coefficients c and d:

$$\begin{aligned}
&c^{(m)}+d^{(m)}\left(Y_{m-1}\Psi_1+\ldots+Y_1\Psi_{m-1}+Y_0\Psi_m\right)=E,\\
&c^{(m)}\sigma\left(X_0\Psi_m^\dagger+X_1\Psi_{m-1}^\dagger+\ldots+X_{m-1}\Psi_1^\dagger\right)+d^{(m)}=0.
\end{aligned}$$

Here E is 3×3 unit matrix. The solution to this system is:

$$c^{(m)}=\left(E-P\hat{P}\right)^{-1},\ d^{(m)}=-c^{(m)}P. \tag{14}$$

The solution to a similar system for $\hat{c},\hat{d}$ has the same form:

$$\hat{c}^{(m)}=\left(E-\hat{P}P\right)^{-1},\ \hat{d}^{(m)}=-\hat{c}^{(m)}\hat{P}. \tag{15}$$

4. Second order accuracy Block TIB algorithm

The second order of accuracy spoils the Toeplitz structure of the system GLME block matrix G. In particular, the trapezoid formula, which provides the second order of accuracy, leads to the change in the first and last block columns of the matrix, as well as the blocks of its main diagonal. The non-Toeplitz structure formally does not allow the use of effective Levinson-like algorithms for inverting the system block Toeplitz matrix.

We proposed the basic idea of solving the system of equations for the second order of accuracy in [10]. This idea uses the transfer of part of the unknowns from the GLME to the right side and excludes the equation for index 0. The system block matrix becomes Toeplitz again, but unknowns appear on the right side of the equations. After the procedure, the system block matrix again takes the Toeplitz form as in (10). However, the diagonal blocks of the system matrix Ψ_0 and the right side have changed:

$$\Psi_0=\begin{pmatrix} 1 & \sigma\frac{h}{2}\Omega_{1;0}^* & \sigma\frac{h}{2}\Omega_{2;0}^* \\ \frac{h}{2}\Omega_{1;0} & 1 & 0 \\ \frac{h}{2}\Omega_{2;0} & 0 & 1 \end{pmatrix},$$

$$r_k^{(m)}=\begin{pmatrix} \sigma\frac{h}{2}(\Omega_{1;m+1-k}\,\mathrm{v}_{1;m}^{(m)}+\Omega_{2;m+1-k}\,\mathrm{v}_{2;m}^{(m)}) \\ -(1+\frac{h}{2}\mathrm{u}_1^{(m)})\Omega_{1;k} \\ -(1+\frac{h}{2}\mathrm{u}_1^{(m)})\Omega_{2;k} \end{pmatrix},$$

$k=1,\ldots,m$. Note that not all unknown elements appear on the right side, but only the first $\mathrm{u}_1^{(m)}$ and two last elements $\mathrm{v}_{1;m}^{(m)},\mathrm{v}_{2;m}^{(m)}$. The knowledge of the fundamental properties of the inverse block Toeplitz matrix makes us possible to find an efficient TIB calculation algorithm with the second order of approximation accuracy. Multiplying the first and last rows of the inverse matrix by the right side of the GLME system, we get a pair of equations for a pair of unknowns - for the first and for the last element of the block vector of unknowns. Since we search only for the last element of the vector, as a result, we get a solution of the GLME with the second order of accuracy. This formal scheme becomes more complicated in the case of the block Toeplitz matrices. The vector of unknowns for the block version of the Manakov system comprises columns contains of three elements $(\mathrm{u}_k^{(m)},\mathrm{u}_{1;k}^{(m)},\mathrm{u}_{2;k}^{(m)})^T$. As a result, after multiplying the right side by the first $g_1^{(m)}$ and last $g_m^{(m)}$ block rows of the inverse matrix, we get a system of six equations for the columns $(\mathrm{u}_1^{(m)},\mathrm{v}_{1;1}^{(m)},\mathrm{v}_{2;1}^{(m)})^T$ and $(\mathrm{u}_m^{(m)},\mathrm{v}_{1;m}^{(m)},\mathrm{v}_{2;m}^{(m)})^T$. Recall that at the m th step of the algorithm, we are looking for only the last two elements $v_{1;m}^{(m)},v_{2;m}^{(m)}$. The resulting system of six scalar equations can be represented as a pair of linear systems with 3×3 blocks:

$$\begin{pmatrix}\mathrm{u}_1^{(m)}\\ \mathrm{v}_{1;1}^{(m)}\\ \mathrm{v}_{2;1}^{(m)}\end{pmatrix}=-\frac{F}{2}\begin{pmatrix}\dfrac{2}{h}+\mathrm{u}_1^{(m)}\\ 0\\ 0\end{pmatrix}+\frac{P}{2}\begin{pmatrix}0\\ \mathrm{v}_{1;m}^{(m)}\\ \mathrm{v}_{2;m}^{(m)}\end{pmatrix},$$

$$\begin{pmatrix}\mathrm{u}_m^{(m)}\\ \mathrm{v}_{1;m}^{(m)}\\ \mathrm{v}_{2;m}^{(m)}\end{pmatrix}=-\frac{\hat{P}}{2}\begin{pmatrix}\dfrac{2}{h}+\mathrm{u}_1^{(m)}\\ 0\\ 0\end{pmatrix}+\frac{\hat{F}}{2}\begin{pmatrix}0\\ \mathrm{v}_{1;m}^{(m)}\\ \mathrm{v}_{2;m}^{(m)}\end{pmatrix}. \tag{16}$$

From the six equations of the system (16), we singled out a simpler subsystem of only three equations containing the components $\mathrm{v}_{1;m}^{(m)},\mathrm{v}_{2;m}^{(m)}$ of the potential vector sought at each step. The solution of the system (16) with the second order of accuracy is:

$$\mathrm{v}_{1;m}^{(m)}=\left(\frac{2}{h}\right)\frac{-2\hat{p}_{21}+\hat{f}_{33}\hat{p}_{21}-\hat{f}_{23}\hat{p}_{31}}{2+f_{11}-\hat{f}_{22}-\hat{f}_{33}},$$

$$\mathrm{v}_{2;m}^{(m)}=\left(\frac{2}{h}\right)\frac{-2\hat{p}_{31}+\hat{f}_{22}\hat{p}_{31}-\hat{f}_{32}\hat{p}_{21}}{2+f_{11}-\hat{f}_{22}-\hat{f}_{33}}. \tag{17}$$

The solution (17) completes the block algorithm of the second order of approximation accuracy.

5. Schematic of the second order Block TIB algorithm

The main steps of the second order block TIB algorithm are:

1) For $m=1$ calculate initial value for the auxiliary array: $g_1^{(1)}=(\Psi_0)^{-1}$, and calculate P and $\hat{P}$ 3×3 size matrices: $\hat{P}=g_1^{(1)}\Psi_1$, $P=\sigma g_1^{(1)}\Psi_1^{\dagger}$, and also calculate matrices F and $\hat{F}$: $F=g_1^{(1)}\Psi_1$, $\hat{F}=\sigma g_1^{(1)}\Psi_1^{\dagger}$ (of course, we calculate only those elements of the matrices that are used in the algorithm);
2) Find m th component of the solution vector $q_{1;m}=2v_{1;m}^{(m)}$, $q_{2;m}=2v_{2;m}^{(m)}$, uses (17). This is the output at every step;
3) Calculate the block coefficients $c^{(m)},d^{(m)},\hat{c}^{(m)},\hat{d}^{(m)}$, uses (14) and (15);
4) Determine next step for the auxiliary arrays, i.e. rows $g_1^{(m+1)}$ and $g_{m+1}^{(m+1)}$, uses (13);
5) Find the necessary elements of the new matrices $P,F,\hat{P},\hat{F}$ by calculating the sums (12);
6) Increment m and go to the step 2 while $m<N$.

This schematic corresponds to the working interval $[0, T]$. However, this algorithm is invariant regarding the choice of the working interval and the origin of the coordinates, as are for most equations and algorithms in mathematical physics. When the interval is changed, only the calculation grid changes but the algorithm itself does not change.

6. Numerical simulation

To test the algorithm, we mainly used an exact analytical solution of the Manakov model - the Manakov optical vector soliton. The soliton, as a solution of the GLM equations for the eigenvalue of the discrete spectrum $\lambda=\xi+i\eta$, has the following form:

$$q_1(t)=2\eta e^{-\mathrm{i}(\theta_1+2t\xi)}sech(2\eta(t-t_0))\cos(\phi),$$
$$q_2(t)=2\eta e^{-\mathrm{i}(\theta_2+2t\xi)}sech(2\eta(t-t_0))\sin(\phi), \tag{18}$$

where t_0 sets the position of the soliton center, $\theta_{1,2}$ are the phases of the soliton components, and ϕ is the polarization angle. This soliton is the solution of the ISP for the following kernel of the integral equations of GLM:

$$\Omega_1(t)=2\eta e^{2\eta(t-t_0)-\mathrm{i}(\theta_1+2t\xi)}\cos(\phi),$$
$$\Omega_2(t)=2\eta e^{2\eta(t-t_0)-\mathrm{i}(\theta_2+2t\xi)}\sin(\phi). \tag{19}$$

The Fig. 1 shows the dependence of the total error of restoring the Manakov vector soliton on the number of calculation points on a logarithmic (base 2) scale. The straight line corresponds to the linear fitting of this dependence. This line is described by the line function $y=9.26968-2.00027x$. The second fitting parameter 2.00027 is very close to the value 2. It means that the error drops by a factor of 4 when the number of points doubles, which proves the second order of approximation accuracy of the block TIB algorithm. The ISP for the Manakov vector soliton solution was solved on the interval $t\in[-10, 10]$ for the following set of the soliton parameters: $\xi=-3$, $\eta=1$, $t_0=4$, $\theta_1=-5\pi/6$, $\theta_2=\pi/4$, $\phi=7\pi/9$.

The "flat" solutions of the Zakharov-Shabat system, which are also partial solutions of the Manakov model, were also used for testing the algorithm. In particular, the tests used a secant potential [22], $q_\alpha=A_\alpha sech(t)$, which contains a discrete spectrum, along with a continuous one, depending on the amplitude of A_α of its polarizations. Numerical simulation for this solution also showed the stability and efficiency of the presented block TIB algorithm.

The running time of the described algorithm does not depend on the length of the line but depends quadratically on the size of the discrete grid N, as for the case of scalar TIB. This time, of course, greatly depends on the software used. Its estimate can be obtained by comparing it with the execution time of a calculation using a similar algorithm described in [20]. Preliminary calculations showed that the presented algorithm is not inferior in terms of computation speed to the algorithm described in [20], where the running time depending on N is estimated as $1.35N^2 10^{-5}$ seconds. This estimate is for GNU Fortran program, running under Windows 7, 64-bit, Intel Core i7 Q720, 4 cores, 1.6 GHz base clock, 10 GB RAM.

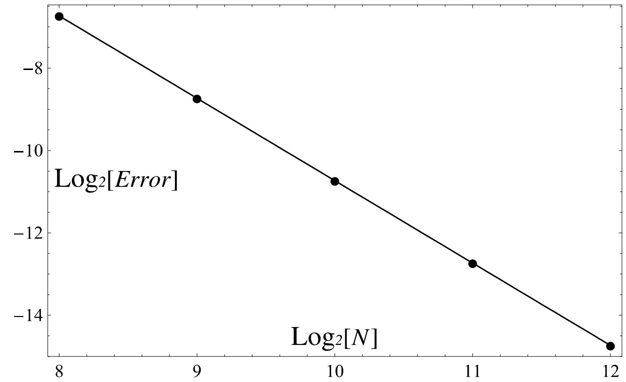

Fig. 1. Inverse scattering problem: Log-Log dependence of the recovery error for the Manakov vector soliton solution on the number of calculation points for the block TIB algorithm

The algorithm turned out to be very convenient for modeling collisions (interactions) of polarized optical vector solitons in the Manakov model. We present here a result of this simulation as an example of the application of the method. The analytical formulas for the 2-soliton solution in the Manakov model are very cumbersome. However, the GLME integral kernel has a relatively simple form: it is the sum of the kernels (19) of the two Manakov vector solitons. Solving the ISP for these kernels using the block TIB algorithm, we numerically find the 2-soliton solution. This solution can be interpreted as a result of a collision of a pair of differently polarized Manakov solitons, if the solitons parameters are properly chosen. In this calculation, the following two sets of data for pairs of soliton kernels were used: $\xi_1 = 0$, $\eta_1 = 0.5$, $t_{1,0} = 0$, $\theta_{1,1} = \pi/6$, $\theta_{1,2} = \pi/6$, $\phi_1 = 0$ and $\xi_2 = 1$, $\eta_2 = 1$, $t_{20} = 1$, $\theta_{2,1} = 2\pi/6$, $\theta_{2,2} = 2\pi/3$, $\phi_2 = \pi/2$. Here the first index is the number of the soliton. Fig. 2 shows the result of a numerical simulation of the collision pattern of two differently polarized Manakov solitons, as an illustration of the algorithm capabilities. This Figure illustrates the possibility of an envelope signal to contain the amplitude, phase and polarization components of information, which theoretically allows the implementation of complex amplitude-frequency-polarization modulation formats for more efficient transmission of information through the optical lines.

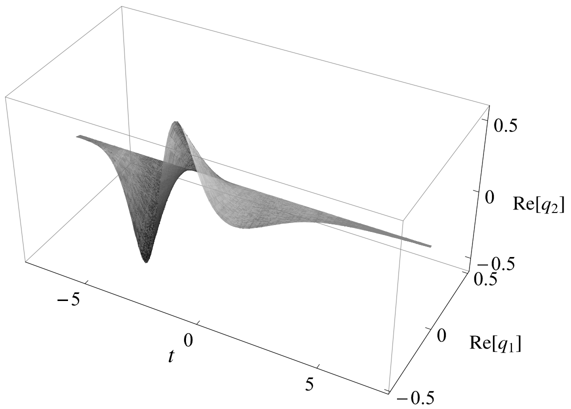

Fig. 2. The algorithm aprobation: simulation of the collision of two differently polarized Manakov solitons

For comparison, Fig. 3 shows the envelope signal intensity ($|q|^2$) distribution in time for the collision of two differently polarized Manakov solitons. In this signal, the phase and polarization components of information are completely lost, and only amplitude modulation of the signal can be used to information transmission. This variant was used at the initial stages of optical information transmission technology.

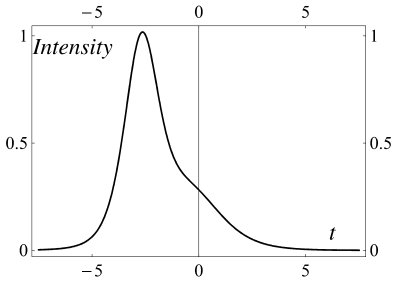

Fig. 3. Envelope signal intensity time distribution for the collision of two differently polarized Manakov solitons

Conclusion

The paper describes an efficient numerical algorithm for solving the inverse spectral scattering problem associated with the Manakov model. The model comprises a pair of coupled Schrödinger equations with second-order dispersion and cubic (Kerr) nonlinearity for two optical polarizations. This model of wave processes simultaneously considers dispersion, nonlinearity, and polarization effects. It is in demand in nonlinear physical optics and is especially perspective for describing optical radiation propagation through the fiber communication lines.

The study of the scattering problems for the Manakov system, associated with the model, leads to a system of linear integral equations - GLME. The numerical solution to the inverse scattering problem based on the inversion of matrices corresponding to the discretized system of GLME, using the described block version of the Levinson-type Toeplitz inner bordering (TIB) algorithm. Numerical tests performed by comparing the results of calculations with the known exact analytical solutions (Manakov vector soliton and Sech-potential) confirm the stability and second order of accuracy of the proposed algorithm. We also include an example of the algorithms' application to simulate the collision of the differently polarized pair of Manakov vector solitons.

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Authors' information

Alexander Evgenievich Chernyavsky, b. 1999, in 2020 graduated from Physical dep. of Novosibirsk State University (NSU) with a degree in "Quantum Optics". Now he is magistrant of the NSU and works as an engineer-programmer at the Institute of Automation and Electrometry SB RAS. The field of scientific interests: quantum optics, the Method of inverse scattering problem, computer simulation. E-mail: *alexander.cher.99@gmail.com*

Leonid Lazarevich Frumin, b. 1953, in 1976 graduated from Physics dep. of Novosibirsk State University. In 1983 he received candidate degree in "Optics" at the Institute of Heatophysics of the SB of USSR Academy of Sciences. In 2005, he defended his doctoral dissertation with degree in the "Physics of Condensed Matter" at Moscow State University. He is leading researcher in the Photonics laboratory of the Institute of Automation and Electrometry SB RAS. He has 85 scientific articles and 3 international patents. The field of scientific interests: photonics, plasmonics, computer simulation, the Method of inverse scattering problem, condensed matter physics. E-mail: *lfrumin@iae.nsk.s*.

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