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UNIVERSAL INVARIANTS OF PARAXIAL OPTICAL BEAMS

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Abstract—For paraxial optical beams propagating in a medium whose parabolic transverse profile of dielectric permeability varies arbitrarily along the beam direction, universal invariants have been determined, i.e. certain integral values that remain valid along the beam axis independently of the particular dependence of the dielectric permeability on the coordinates. The effect of medium nonparabolicity on the invariants is discussed.

It is known that for harmonic wave fields propagating in weakly nonuniform media the Helmholtz equation for the field components in the Leontovich–Fock paraxial approximation [1] leads to a parabolic nonstationary Schroedinger equation [2]. If the z-axis of a rectangular coordinate system $\{x_1, x_2, x_3\}$ is chosen along the direction of propagation, then the parabolic equation assumes the form

$$i\lambda\frac{\partial\psi}{\partial z} = -\frac{\lambda^2}{2n_0}\left[\frac{\partial 2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2}\right] + \frac{n_0^2 - n^2}{2n_0}\psi = -\frac{\lambda^2}{2n_0}\left[\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2}\right] + V\psi, \qquad (1)$$

where $E = 2n_0^{-1/2}\psi \exp\left(\frac{i}{\lambda}\int_0^z n_0(z)\,dz\right)$ is one of the field components, $n_0 = n(0, 0, z)$ is the refractivee index of the medium along the z axis, $k = 2\pi/\lambda = \lambda^{-1}$ is the wave number in vacuum, and ψ is an amplitude varying slowly over a wavelength.

By effecting the transformation $k^{-1} \to h$ and $z \to t$, Eq. (1) is seen to be the quantum-mechanical Schroedinger equation for a unit mass particle moving in the potential field $V = (n_0^2 - n^2)/2n_0$.

The operators of conjugate canonical variables $\hat{x}_i \to x_i$ and $\hat{p}_1 - i\lambda\partial/\partial x_i$ satisfy the usual commutation relations

$$[\hat{x}_i, \hat{p}_j] = i\lambda\delta_{ij}, \quad i, j = 1, 2).$$

The formal analogy [2] of the parabolic equation to the Schroedinger equation allows one to utilize methods developed in quantum mechanics to solve wave propagation problems in nonuniform media.

It has been shown in [3] that for certain classes of Hamiltonians, in particular, for any nonuniform multidimensional quadratic form of operators whose commutators are c-numbers, there exist conserved quantities (along the beam axis, in the present case) which depend only on the initial state of the system and the form of the commutation relations, and not on the coefficients of the corresponding quadratic or linear forms. These quantities were termed universal invariants, by analogy with the Poincaré–Cartan universal invariants in classical mechanics. In order to obtain similar invariants for the propagation of paraxial beams in optical media we consider a system of four operators

$$\hat{Q}_1 = \hat{x}, \qquad \hat{Q}_2 = \hat{y}, \qquad \hat{Q}_3 = \left(-i\lambda\frac{\partial}{\partial x}\right), \qquad \hat{Q}_4 = \left(-i\lambda\frac{\partial}{\partial y}\right).$$

The specific problem addressed here differs from the one dealt with in [3], in that instead of considering a system with any number of degrees of freedom we limit ourselves to a set of two observables. But in return we obtain the explicit form of the invariants.

If the dielectric permeability of the medium, n^2, is a quadratic or linear function of the transverse coordinates x, y (and an arbitrary function of z, the coordinate along the beam), then in the Heisenberg representation the operators $\hat{Q}_\alpha(z)$ depend linearly on $\hat{Q}_\alpha(0)$:

$$\hat{Q}_\alpha(z) = \Lambda_{\alpha\beta}(z)\hat{Q}_\beta(0) + \delta_\alpha(z). \qquad (2)$$

We introduce the centred second-order moments

$$Q_{\alpha\beta} = \overline{Q_\alpha Q_\beta} = \langle \tfrac{1}{2}(\hat{Q}_\alpha \hat{Q}_\beta + \hat{Q}_\beta \hat{Q}_\alpha) \rangle - \langle \hat{Q}_\alpha \rangle \langle \hat{Q}_\beta \rangle, \tag{3}$$

where the angular brackets $\langle \cdots \rangle$ denote the expected value of the operators.

As a consequence of (2), the second-order moments satisfy the following relationship:

$$Q_{\alpha\beta}(z) = \Lambda_{\alpha\mu}(z) Q_{\mu\nu}(0) \Lambda_{\beta\nu}(z). \tag{4}$$

Since the transformation (2) preserves the commutators, we have the identity

$$\Lambda(z)\Sigma\tilde{\Lambda}(z) = \Sigma, \qquad \Sigma = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix}, \qquad \Lambda = \|\Lambda_{\alpha\beta}\| \tag{5}$$

from which it follows that $\det \Lambda(z) = 1$. Writing (4) in matrix form

$$Q(z) = \|Q_{\alpha\beta}(z)\| = \Lambda(z) Q(0) \tilde{\Lambda}(z) \tag{6}$$

and comparing with (5), we obtain for any parameter μ, assuming the matrix Σ is nondegenerate, the identity

$$D(\mu) = \det[\Sigma Q(z) - \mu E] = \sum_{m=0}^{N} D_{2m}^{(N)} \mu^{2m}. \tag{7}$$

The quantities $D^{(N)}$ are universal invariants, since they are conserved as the beam propagates along the z-axis and do not depend on the detailed form of the coefficients in the quadratic dependence of the dielectric permeability on the coordinates x, y. Since the matrix Q is symmetric and Σ is antisymmetric, the series in (7) contains only even powers of μ. N is the number of transverse coordinates on which the amplitude will depend (either 1 or 2).

For a dielectric permeability having a general quadratic dependence on the coordinates, the D_{2m} are given by the following expressions in the two-dimensional case:

$$D_2 = -2(\overline{xp_y})(\overline{yp_x}) - (\overline{yp_y})^2 - (\overline{xp_x})^2 + (\overline{p_y^2})(\overline{y^2}) + (\overline{p_x^2})(\overline{x^2}) + 2(\overline{xy})(\overline{p_x p_y}), \tag{8}$$

$$\begin{aligned} D_0 = {} & (\overline{xp_x})^2(\overline{yp_y})^2 + (\overline{xp_y})^2(\overline{yp_x})^2 - 2(\overline{xy})(\overline{p_x p_y})(\overline{xp_x})(\overline{yp_y}) - 2(\overline{xp_x})(\overline{yp_y})(\overline{xp_y})(\overline{yp_x}) \\ & + 2(\overline{xy})(\overline{p_x^2})(\overline{xp_y})(\overline{yp_y}) - (\overline{y^2})(\overline{p_x^2})(\overline{xp_y})^2 + 2(\overline{xp_x})(\overline{xp_y})(\overline{p_x p_y})(\overline{y^2}) + 2(\overline{xp_x})(\overline{p_y^2})(\overline{xy})(\overline{yp_x}) \\ & - (\overline{y^2})(\overline{p_y^2})(\overline{xp_x})^2 - 2(\overline{xy})(\overline{p_x p_y})(\overline{xp_y})(\overline{yp_x}) + 2(\overline{x^2})(\overline{yp_x})(\overline{yp_y})(\overline{p_x p_y}) - (\overline{x^2})(\overline{p_x^2})(\overline{yp_y})^2 \\ & - (\overline{x^2})(\overline{p_y^2})(\overline{yp_x})^2 + (\overline{x^2})(\overline{y^2})(\overline{p_x^2})(\overline{p_y^2}) - (\overline{p_x p_y})^2(\overline{x^2})(\overline{y^2}) - (\overline{xy})^2(\overline{p_x^2})(\overline{p_y^2}) + (\overline{xy})^2(\overline{p_x p_y})^2. \end{aligned} \tag{9}$$

It is of particular interest to consider the special case of an axially symmetric medium, when n^2 depends only on $x^2 + y^2$ (as in a fibre light guide). Then if the matrix $Q(z)$ at $z = 0$ is invariant with respect to a transformation of simultaneous rotation by an angle Φ in the (x, y) and (p_x, p_y) planes, it will remain invariant with respect to this transformation as the beam propagates along the z-axis. In this case the $D_{2m}^{(N)}$ become (to within a constant)

$$D_2^{(2)} = (\overline{xp_y})^2 - (\overline{xp_x})^2 + (\overline{x^2})(\overline{p_x^2}), \tag{10}$$

$$D_0^{(2)} = [(\overline{xp_x})^2 + (\overline{xp_y})^2 - (\overline{x^2})(\overline{p_x^2})]^2. \tag{11}$$

In the one-dimensional case ($N = 1$) (a planar light guide) the universal invariant is

$$D_0^{(1)} = (\overline{x^2})(\overline{p_x^2}) - (\overline{xp_x})^2. \tag{12}$$

In the paraxial approximation not just the Helmholtz equation but the complete wave equation becomes similar to a Schroedinger equation:

$$i\lambda\!\!\!^{-} \frac{\partial \psi}{\partial z} = \frac{\lambda\!\!\!^{-2}}{2n_0}\left[\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right] + \frac{n_0}{2}\psi. \tag{13}$$

Consequently, for any (sufficiently smooth) dependence of the refractive index n on the longitudinal coordinate z, there will be universal invariants comprising time-like moments of the type

$$\overline{(t^2)} = \int \psi^*(x, y, t) t^2 \psi(x, y, t)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t, \qquad \overline{(p_t^2)} = -\lambda^2 \int \psi^* \frac{\partial^2}{\partial t^2} \psi(x, y, t)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t, \quad \left(\hat{p}_t = \mathrm{i}\lambda \frac{\partial}{\partial t}\right),$$

describing space and time limited beams:

$$D_2^{(1)} = \overline{(t^2)}\,\overline{(p_t^2)} + \overline{(x^2)}\,\overline{(p_x^2)} - \overline{(tp_t)}^2 - \overline{(xp_x)}^2 - 2\overline{(xt)}\,\overline{(p_xp_t)} + 2\overline{(xp_t)}\,\overline{(tp_x)}, \tag{14}$$

$$\begin{aligned} D_0^{(1)} = {} & \overline{(x^2)}\,\overline{(t^2)}\,\overline{(p_x^2)}\,\overline{(p_t^2)} + \overline{(xt)}^2\overline{(p_xp_t)}^2 + \overline{(xp_x)}^2\overline{(tp_t)}^2 + \overline{(xp_t)}^2\overline{(tp_x)}^2 - \overline{(p_x^2)}\,\overline{(p_t^2)}\,\overline{(xt)}^2 - \overline{(x^2)}\,\overline{(t^2)}\,\overline{(p_xp_t)}^2 \\ & - \overline{(x^2)}\,\overline{(p_x^2)}\,\overline{(tp_t)}^2 - \overline{(t^2)}\,\overline{(p_x^2)}\,\overline{(xp_t)}^2 - \overline{(p_t^2)}\,\overline{(t^2)}\,\overline{(xp_x)}^2 - \overline{(x^2)}\,\overline{(p_t^2)}\,\overline{(tp_x)}^2 + 2\overline{(p_t^2)}\,\overline{(xp_x)}\,\overline{(tp_x)}\,\overline{(xt)} \\ & + 2\overline{(t^2)}\,\overline{(xp_x)}\,\overline{(p_xp_t)}\,\overline{(xp_t)} + 2\overline{(p_x^2)}\,\overline{(xt)}\,\overline{(xp_t)}\,\overline{(tp_x)} + 2\overline{(x^2)}\,\overline{(tp_x)}\,\overline{(p_xp_t)}\,\overline{(tp_t)} - 2\overline{(xp_x)}\,\overline{(tp_t)}\,\overline{(tp_x)}\,\overline{(xp_t)} \\ & - 2\overline{(xp_x)}\,\overline{(tp_t)}\,\overline{(xt)}\,\overline{(p_xp_t)} - 2\overline{(xt)}\,\overline{(p_xp_t)}\,\overline{(tp_x)}\,\overline{(xp_t)}, \end{aligned} \tag{15}$$

$$D_6^{(2)} = 1$$

$$\begin{aligned} D_4^{(2)} = {} & 2\overline{(xy)}\,\overline{(p_xp_y)} - \overline{(tp_t)}^2 - \overline{(yp_y)}^2 - 2\overline{(xt)}\,\overline{(p_xp_t)} + \overline{(x^2)}\,\overline{(p_x^2)} + \overline{(y^2)}\,\overline{(p_y^2)} + \overline{(t^2)}\,\overline{(p_t^2)} - \overline{(xp_x)}^2 \\ & - 2\overline{(yt)}\,\overline{(p_yp_t)} - 2\overline{(xp_y)}\,\overline{(yp_x)} + 2\overline{(xp_t)}\,\overline{(tp_x)} + 2\overline{(yp_t)}\,\overline{(tp_y)}. \end{aligned} \tag{16}$$

We introduce a Gaussian density matrix of an axially-symmetric optical beam propagating in an axially-symmetric medium.

$$\rho = N \exp(-a(x_1^2 + y_1^2) - a^*(x_2^2 + y_2^2) + 2b(x_1x_2 + y_1y_2) - fx_1y_2 - f^*x_2y_1).$$

Calculating the dispersion and substituting it into the universal invariants we obtain the following relations:

$$\frac{(a + a^*)^2 - f^2}{(a + a^*)^2 - 4b^2} = \text{const}, \qquad \frac{4b^2 + f^2}{(a + a^*)^2 - 4b^2} = \text{const},$$

$$\frac{4b^2 + f^2}{(a + a^*)^2 - f^2} = \text{const}, \qquad \frac{b}{a + a^*} = \text{const},$$

$$\frac{2}{\lambda^2}[D_0^{(2)} - D_2^{(2)}] = \frac{f}{(a + a^* - 2b)} = \text{const}.$$

Physically these equations represent the conservation of the correlation radius to beam width ratio, as well as of the "impulse moment" $\langle xp_y - yp_x \rangle$.

Lastly, we discuss the conservation of the above invariants for a nonquadratic medium, taking a one-dimensional example with an effective potential $V(x)$ (cf. Eq. (1));

$$V(x) = \frac{w^2}{2} x^2 + \sum_{n \geqslant 3} A_n x^n / n.$$

If $A_n \equiv 0$ we have the invariant (12). If the A_n are non-zero we readily obtain

$$\frac{\mathrm{d}D}{\mathrm{d}z} = \sum_{n \geqslant 3} A_n [\langle x^n \rangle \langle px + xp \rangle - \langle x^2 \rangle \langle px^{n-1} + x^{n-1}p \rangle]. \tag{17}$$

This clearly shows that in general D depends on z whenever $A_n \neq 0$. However, if the anharmonic terms are small, i.e. $A_n \to 0$, then in calculating $D(z)$ we may use the zero-order values for the higher moments entering the right-hand side of (17) (i.e. computed with $A_n = 0$). In that case $D(z)$ will oscillate near the initial value with an amplitude of order A_n as $A_n \to 0$. However, it is significant that for a certain class of initial states this amplitude is of higher order smallness than A_n. For example, if only A_3 and A_4 are nonzero, this situation obtains for a Gaussian initial state (when the mutual coherence function is an exponential of a quadratic form), since in zero order a Gaussian state stays Gaussian with zero first order means, provided it started out that way initially. Then the coefficient A_3 becomes zero, since for a Gaussian state all odd moments vanish if they are vanishing in first order, whereas A_4 vanishes on account of the well-known relation for Gaussian distributions:

$$\langle x^4\rangle = 3(\langle x^2\rangle)^2, \qquad \langle px^3 + x^3p\rangle = 3\langle x^2\rangle\langle xp + px\rangle.$$

It is also interesting that for non-Gaussian states, and when $A_3 \neq 0$ (in zero order), the function $D(z)$ will again oscillate around the initial value $D(0)$:

$$\begin{aligned} D &= D(0) + \Delta D \\ &= AB - C^2 - 3A_3\frac{1}{\gamma}\left[\frac{\sin 3\gamma z}{3}\left(\frac{CM}{4} + \frac{CL}{4\gamma^2} + \frac{AN}{4\gamma^2} - \frac{AP}{4} - \frac{BP}{2\gamma^2}\right)\right. \\ &\quad + \sin\gamma z\left(\frac{3CM}{4} - \frac{CL}{4\gamma^2} - \frac{AN}{4\gamma^2} + \frac{BP}{2\gamma^2} - \frac{3AP}{4}\right) \\ &\quad + \frac{1 - \cos 3\gamma z}{3}\left(\frac{CP}{4\gamma} + \frac{CN}{4\gamma^3} - \frac{AL}{2\gamma} - \frac{BL}{4\gamma^3} + \frac{BM}{4\gamma}\right) \\ &\quad \left. + (1 - \cos\gamma z)\left(-\frac{3CP}{4\gamma} + \frac{3CN}{4\gamma^3} - \frac{AL}{2\gamma} + \frac{3BL}{4\gamma^3} + \frac{BM}{4\gamma}\right)\right], \end{aligned}$$

where

$$A = (\overline{x^2})|_{z=0}, \quad B = (\overline{p^2})|_{z=0}, \quad C = (\overline{xp})|_{z=0}, \quad M = (\overline{x^3})|_{z=0},$$
$$P = (\overline{px^2})|_{z=0}, \quad L = (\overline{xp^2})|_{z=0}, \quad N = (\overline{p^3})|_{z=0}.$$

If only A_4 is nonzero, $D(z)$ will increase:

$$\begin{aligned} D(z) &= AB - C^2 + \frac{1}{4\gamma^2}A_4\left[\frac{\sin 4\gamma z}{4}\left(\frac{aC\gamma}{8} - \frac{Cb}{8\gamma^2} + \frac{Bk}{8\gamma^3} - \frac{3Bn}{8\gamma} - \frac{An\gamma}{8} + \frac{3Ak}{8\gamma}\right)\right. \\ &\quad - \frac{\cos 4\gamma z}{4}\left(\frac{Ck}{4\gamma^2} - \frac{Cn}{4} - \frac{aB}{8} + \frac{3Bm}{8\gamma^2} + \frac{Ab}{8\gamma^2} - \frac{3Am}{8}\right) \\ &\quad - \frac{\cos 2\gamma z}{2}\left(\frac{Cn}{2} - \frac{Ck}{2\gamma^2} + \frac{aB}{4} - \frac{3Bm}{4\gamma^2} - \frac{Ab}{4\gamma^2} + \frac{3Am}{4}\right) \\ &\quad + \frac{3}{2}\gamma z\left(\frac{aC\gamma}{4} - \frac{Cb}{4\gamma^3} + \frac{Bk}{4\gamma^3} + \frac{Bn}{4\gamma} - \frac{An\gamma}{4} - \frac{Ak}{4\gamma}\right) \\ &\quad \left. + \frac{\sin 2\gamma z}{2}\left(\frac{aC\gamma}{2} + \frac{Cb}{2\gamma^3} - \frac{Bk}{2\gamma^3} - \frac{An\gamma}{2}\right)\right], \end{aligned}$$

where

$$n = (\overline{x^3p})|_{z=0}, \quad k = (\overline{p^3x})|_{z=0}, \quad a = (\overline{x^4})|_{z=0}, \quad b = (\overline{p^4})|_{z=0}, \quad m = (\overline{x^2p^2})|_{z=0}.$$

In addition to the above invariants associated with second-order moments, one may construct more complicated universal invariants expressed in terms of fourth-order moments [4].

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