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SYNTHESIZED PHASE ELEMENTS FOR INTEGRAL TRANSFORMATIONS OF COHERENT OPTICAL FIELDS

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Abstract—Those reports of the authors that have been devoted to computer generation of phase optical elements for various integral transforms of coherent optical fields are reviewed. The topics covered include Bessel transforms of high order, phase diffraction gratings with complex spectrum, and curvilinear transformations of coordinates.

INTRODUCTION

The present paper reviews a series of publications on the subject [2, 3, 5, 6, 10, 16].

Recent years have seen a wide application of coherent optical radiation where considerations of physics and technology require fields of complex and controlled structure. These fields are above all required in optical instrumentation, optical data-processing, fibre-optical communications and integral optics. Analysis of coherent field applications reveals a trend from simple optics toward complex and controlled emission patterns.

A principal method of flexible control of the structure of coherent optical wavefields in different media consists in decomposing the field over various full systems of functions (modes). Such integral transformations represent the field as a spectrum of generalized spatial frequencies and serve as a basis for methods of spatial filtration. Thus, investigations of the optical Fourier transformation have led to a breakthrough in coherent optics and its applications.

The importance of phase in signal processing and the possibilities of phase control of optical signals has been stressed by many works [9, 46, 82, 86]. Even apart from energy considerations, phase control of coherent fields is advantageous in that the optical system may incorporate reflecting elements (including adaptive mirrors) with appropriate achromatic properties and stability at high power [30]. The development of phase methods of control of coherent fields is however hampered by the inadequate state of high-resolution optics for distortion-free imagery and technology of complex phase profile [70, 76]. The complexity of the associated mathematical problems only aggravates the situation. Further progress in this field depends on adaptive optics technology and on handling a number of new problems of field control such as nonlinear problems related to phase control, complex two-dimensional problems, and invention of new complete systems of functions. Development of these methods should bring new life into the methods of adaptive field control, but at the initial state of the evolution they may also be used independently.

The two-dimensional Fourier transform is fundamental to and most widely used in present-day coherent optics. Its applications have been amply reported. These applications are above all based on frequency filtering of two-dimensional signals [56, 58, 66, 67] (including matched holographic filtering for pattern recognition).

Frequency filtering suppresses certain frequencies in the spectrum of the signal and singles out its useful part. Another method of signal processing involves using phase filters in both frequency and spatial domains.

Phase filters do not cause loss of information but allow one to control the signal structure, after transformations, and the like, imposed on the signal. One example of this approach is a Mellin (essentially one-dimensional) transform obtained from the one-dimensional Fourier transform by changing the coordinate with the aid of a one-dimensional phase filter. The transformation can be done in two coordinates simultaneously. Other examples will be considered in the ensuing sections.

1. BESSEL OPTICS

The term Bessel optics has been coined by analogy with Fourier-optics [25] because the integral transformations with a Bessel function as the key operation plays the same part as a Fourier transformation in Fourier-optics, and the Bessel transform of zero order coincides with the Fourier transform (for fields with circular symmetry).

In the optical implementation of arbitrary-order Bessel transformations the first consideration emphasizes its utility for axially symmetric systems such as laser resonators and optical multimode fibres [68]. The approach proposed in this paper is based on aligning systems with spatial filtering with coordinate-transformation systems. The designs are very complicated, and the output signals are unsuitable for direct launching into an optical channel. These systems would be more like analog computers than optical processors, and the very principle of joint transformation of coordinates in the integral transformation system is not quite correct physically since the coordinate transformation [71] is based on geometrical optics which leads to strong and unremovable distortions in the resulting transformation. The diffraction approximation is a better alternative.

1.1. *Calculation of a phase filter for expansion of an axially symmetric coherent optical field in Bessel modes of given order*

The series expansion in Bessel functions and the Fourier–Bessel integral are well reported theories [43, 58, 61].

In the space of functions with a finite energy for each $N = 0, 1, 2$, etc., the Bessel functions $J_N(c_k x)$ make up a full orthogonal system.

The three first Bessel modes contain 99.8% of Gaussian beam energy, which indicates the efficiency of expansions in a Fourier–Bessel series (an expansion of this accuracy in a Fourier series of two variables would require up to 20 terms). This result follows directly from the symmetry of the problem and its unidimensionality. Indeed, the one-dimensional expansion requires m coefficients against m^2 for the two-dimensional expansion, of the same accuracy.

The Bessel transform

$$f(r) \rightarrow F(R),$$

$$F(R) = \int_0^\infty r f(r) J_N(rL)\,\mathrm{d}r$$

is a self-reciprocal integral transformation and coincides with its inverse. It carries Bessel harmonics into delta-functions, so that filtering occurs in the spatial frequency domain just as in Fourier-optics.

Moreover, because of the final aperture of the optical system (windowing), the transformation yields rings of final width, rather than delta-functions, their structure being determined by Lommel integrals. Numerical plotting of these quantities shows that these curves are similar to sinc-functions in Fourier-optics. More specifically, they are narrow rings having a maximum where delta-functions should be, the intensity rapidly decaying towards the edge, and some diffraction oscillations at the edges. Bessel transformations of higher order convert Gaussian beams into annular beams and store their original structure.

To take an Nth-order Bessel transform of a one-dimensional signal $f(r)$ with the aid of a two-dimensional optical system, a two-dimensional signal must be rotationally symmetric

$$f(x, y) = f(r), \qquad r^2 = x^2 + y^2.$$

An optical element (Bessel-element of order N) whose surface consists of N segments of a helicoid of total height of $\lambda(n-1)$ is placed in the input plane of a Fourier-transform system. Thus, the complex transmission function of the Nth-order Bessel element is selected equal to $e^{iN\theta}$ where θ is the polar angle in the plane of the element. The Fourier–Bessel transform is obtained by taking a two-dimensional Fourier transform of the signal $e^{iN\theta}$.

The signal $e^{iN\theta}F(R)$ obtained in the output plane has the same phase distortion that was introduced at the input and this can be eliminated by a Bessel element of the order $-N$, that is, the same element but turned by 180°.

In a Bessel-element, the phase increases counterclockwise along circles concentric with the origin (viewed in the direction of rays along the optical axis). If the element is turned back to front, the phase will increase clockwise. The complex transmission function of the turned elements is $e^{-iN\theta}$ rather than $e^{iN\theta}$ that is, conjugate to $e^{iN\theta}$. Therefore, it may be used to suppress the phase of $e^{iN\theta}F(R)$ (though for arbitrary phase elements, this does not hold).

If an amplitude detector is placed in the output plane, the second Bessel-element is no longer necessary. Thus, to get the Bessel transform of the optical field $f(x, y) = f(r)$, it suffices to insert in the input plane a quadratic phase filter (lens) that performs the Fourier-transform, and place two phase filters $e^{iN\theta}$ and $e^{-iN\theta}$, respectively, in the input and output planes.

The coefficients of the series may be determined by performing the Nth-order Bessel transform in which $J_N(b_x x) \leftrightarrow \delta(R - b_k)$, and then measuring the intensity of the resulting rings by cutting them off with annular apertures or using a silicon photodetector with annular elements [60].

It should be noted that the set of generic components of Bessel-optics differs from that of Fourier-optics by the presence of phase filters with the complex transmission $e^{\pm iN\theta}$. Thus, a practical conversion of Fourier-optics to Bessel-optics is fairly easy.

It is instructive to note that these same elements yield basic solutions of the two-dimensional Helmholtz equation in the form $e^{iN\theta}J_N(cr)$, so that these elements may be employed for solving different problems of waveguiding [1, 4, 41, 50, 64].

1.2. *Effect of discretization and quantization on performance of a synthesized filter*

Irrespective of the method of coding and recording of a filter, discretization of its phase function can be described as two consecutive operations [25, 58].

The first operation, multiplication by a comb of delta functions, produces new images that form a grating of period Z/d in the frequency plane (Z is the focal parameter, the product of the wavelength of light by the focal length of the Fourier lens).

The additional images due to diffraction do not affect the efficiency of the synthesized Bessel-element, since all discretization-induced sidelobes are outside the working range of spatial frequencies (from $-1/2d$ to $+1/2d$ in compliance with the Nyquist limit).

Discretization entails: (1) energy losses into the higher orders of diffraction at the grating (up to 30%); (2) limited range of spatial frequencies of the processed signal (Nyquist limit).

Finally, discretization breaks down the phase function of the Bessel-element that has a singularity at the centre ($r = 0$), which covers a radius of about Nd. This manifests itself as a scattering of signal energy at pseudo-random angles in the ring of this radius.

The effect of an M-level quantization of the phase function on the performance of a Bessel-element is harder to estimate.

The most accurate results have been obtained by treating quantization as nonlinear phase distortion [78, 79] obeying the same law over the entire element (the whole range of phase values is divided into M sections and the average value over each of them is taken).

Denote the phase by p, and the quantized phase by $P(p)$. Since $P(p)$ is periodic it can be expanded into a Fourier series. Expanding the rectangular pulses

$$\mathrm{rect}(x) = \begin{cases} 1, & |x| \leqslant 1/2 \\ 0, & |x| > 1/2 \end{cases}$$

in a Fourier series and summing up yields

$$e^{iP(p)} = \sum_{k=-\infty}^{+\infty} \mathrm{sinc}\left(\frac{1}{M}\right) e^{i(kM+1)p} \frac{(-1)^k}{kM+1}.$$

This equation shows that, in addition to the transform of given order N for $k = 0$, the quantization of the phase function of a Bessel element also gives rise to Bessel transforms of multiple orders $(kM+1)N$, $k = \pm 1, \pm 2, \pm 3, \ldots$, that sum up coherently. The transform corresponding to $k = -1$ has the highest energy. At $M = 0$ (binary quantization) the energies for $k = 0$ and -1 are equal, that is, Bessel transforms of orders N and $-N$ add coherently. This results in a periodic modulation of the output signal by the angle θ, close in form to $\cos^2(N\theta)$.

This formula also yields an estimate of the total energy of quantization-induced error, in fractions of the signal energy over the total energy.

ε	0.60	0.19	0.05	0.01	0.003	0.0008
M	2	4	8	16	32	64

Therefore, phase quantization gives rise to "structured" error signals at high generalized spatial frequencies (radii in the output plane), for after the transformation of order N the average radius of the signal increases approximately in direct proportion to $\sqrt{N}$. This suggests that at large M the quantization errors are not only low in energy, but their overlap of the useful output signal is insignificant so that the appropriate number of phase quantization levels should be small.

All this relates to the quantization errors for Bessel-elements made as elements of plane optics, that is, as a phase relief.

When these elements are prepared as artificial holograms, i.e. by methods of digital holography with application of the spatial carrier, the expression $p + ax + by$ should be substituted for p in the formula for phase; here (a, b) is the vector of the spatial carrier.

This indicates that quantization-induced Bessel transforms of multiple order do not coherently overlap on the main lobe, but appear in higher orders of diffraction.

If a carrier is used, a maximum permissible spatial frequency is required (the limitation is that the phase variation period $p + ax + by$ be everywhere below $2d$ in both x and y) to separate the working order of diffraction from higher orders. This in turn imposes limitations (2–4 times) on the maximum spatial frequencies of the phase of the Bessel-element, that is, it reduces the permissible order N of transformation and enlarges the radius of the circle in which the phase function decays, thus limiting the spatial range of processed signals, from below.

1.3. *Spectrum distribution obtained by synthesized filter*

As Bessel elements we have used photomasks prepared as amplitude holograms (mostly binary masks [15]) on photographic film (Fig. 1). We did not take into account either the phase noise of the substrate and emulsion, or nonlinear characteristics of the record.

The optical system was arranged of standard elements from a UIG-2 set (400-mm lens) and a He–Ne 4-mW laser. The 24-order Bessel transform of a simple input signal (circular-aperture uniformly illuminated by a plane wavefront) was observed in the +1 and −1 orders of diffraction. Inaccuracy of the manual positioning of the aperture about the optical axis was observed as some

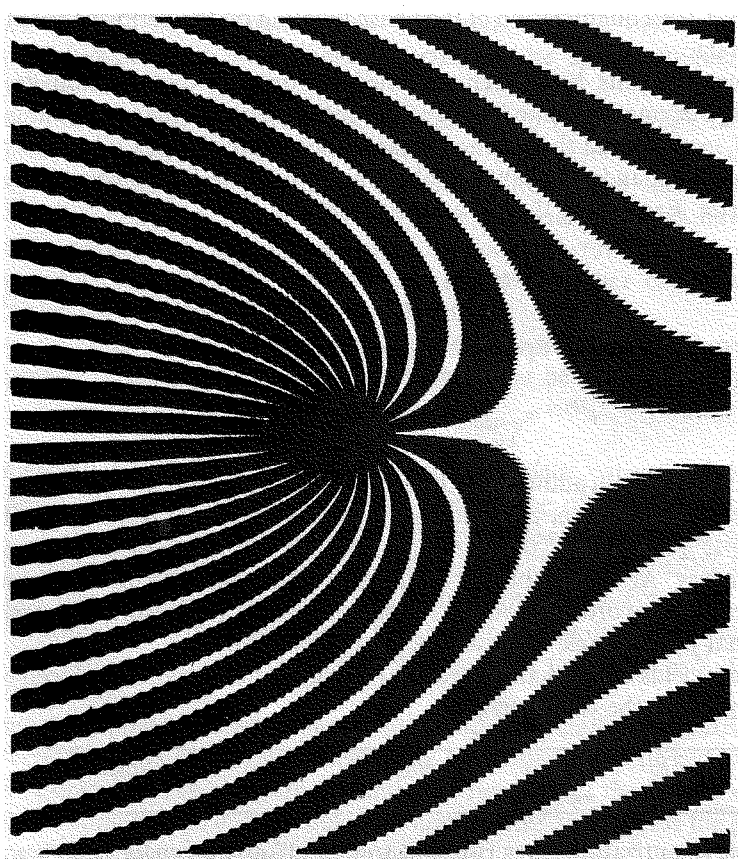

Fig. 1. A 24th-order Bessel element with a carrier.

asymmetry of the output signal. The size of the dark spot at the centre of the spectral distribution corresponds to the figure obtained by a computer.

1.4. *Excitation of Bessel modes of given order in step-index optical fibre*

As noted above, the Bessel transform of any order coincides with its inverse, therefore it can be used for mode synthesis, that is for synthesis of a coherent field being a given combination of Bessel modes.

Excitation of each mode makes good use of the fact that the Bessel transform of the delta-function (corresponding physically to a thin annular slit centred on the origin) produces in the output plane a mode $e^{iN\theta}J_N(ar)$, the mode number a being proportional to the slit radius. It is important to emphasize that by the amplitude distribution and the form of the wavefront this mode is a mode of a step-index optical fibre everywhere except for a thin transition layer from the core to the cladding. Therefore, this mode can be excited in the fibre almost perfectly (noise level below one per cent).

Simultaneous excitation of a few modes is also feasible with the aid of a plate with several concentric annular slits. Such plates are manufactured in the form of mirror film of metallic chrome deposited on a glass substrate. The slit pattern is etched on the film after a plot drawn by the circular image generator used for plotting diffraction lens masks [62]. This technology results in rings as narrow as 1 μm spaced at 3 μm. These plates can also be used as spatial filters of Bessel optical systems to suppress certain modes or to measure the energy of fixed modes in an axially symmetric field of coherent radiation.

If a plane wave is normally incident on an opaque aperture with one or more thin slits, most of the energy fails to pass the slits and consequently is not used in mode excitation. But because a chromic film has a high reflection coefficient, this kind of plate with annular slits may be used in place of the output semi-transparent plane mirror of a standard semi-confocal resonator of the mode excitation laser. The entire output of this resonator, less the usual diffraction losses, will then pass through the annular slits and the whole energy of lasing will go into mode excitation.

Thus a modified resonator and a Bessel system form the basis for a simple and efficient device for the excitation of Bessel modes of given order in step-indexing fibres.

2. PHASE OPTICAL ELEMENTS WITH GIVEN RADIATION PATTERN

The optical Fourier transform separates the spectral components of a coherent field (plane waves) in space, and by focusing the energy of each component into a spot it allows their intensities to be measured individually; that is, it allows their space-frequency filtering. The inverse of this problem is to obtain a given intensity distribution of a coherent plane wave field by controlling its phase (without energy losses). This problem is a typical nonlinear inverse problem of optics. Its formulation is similar to that of ill-posed problems of mathematical physics [32, 53, 57]. In particular, the set of phase functions over which the solution is sought, is a compact set. In effect, the problem boils down to the control of the amplitude radiation pattern by optimizing the phase function of the modulation element.

Control of the radiation pattern (RP) of synthesized phase optical elements is important in many applications in holography, integral optics, optical data processing, focusing of coherent radiation, optical tomography, and other fields of optics. Substantial progress in solving this type of problem has been obtained in antenna theory [7, 8, 27, 28, 37].

In optical problems, control of radiation pattern is of no less importance, and the range of problems is even wider [9, 22, 25, 45, 52]. We shall consider a problem formulation different from that of RP synthesis in antenna theory. To be specific, edge waves may be neglected here and the objective is to evaluate the possibilities for manipulation of RP in a wide range, rather than to increase the directivity coefficient or to reduce sidelobes. In addition, the accuracy of implementation of phase functions in the optical range is much less than for antennae, therefore obtaining a radiation pattern of extremely high directivity (with 30–40 dB suppression of sidelobes) would be irrational.

Likewise RP evaluations accurate to tenths of a per cent is sufficient for optical applications, so that the common Fresnel and Frauenhofer approximations can be used.

In the pure form, an optical element with a given radiation pattern is, for example, a phase diffraction grating with given energy distribution in diffraction orders. When used in tandem with other synthesized and traditional phase optical elements, this component produces a number of devices of value, beam splitters, multifocus lenses and image multipliers [62, 80], multichannel filters, focusators [30], and such.

A multichannel system with such diffraction gratings as beam splitters was used to build an optical system for pulse, three-dimensional and plasma tomography [5, 6].

Computer-synthesized phase diffraction elements with complicated radiation patterns have been used in laser power meters that divert a given small fraction of energy into the first order of diffraction. They have also been exploited for a two-dimensional transformation of a Gaussian beam into an almost uniform beam, and for multichannel beam splitting of phase quantization (for photolithographic etching).

The experimental part of this report is devoted primarily to one-dimensional phase elements of the IR range. It is based on a large number (several hundreds) of computer simulations. We choose phase diffraction gratings for our experimentation, because they afford simple and reliable measurements of the radiation pattern without the employment of complex specialized equipment.

2.1. *Calculation of phase-function for specified radiation pattern*

Suppose that a plane wavefront of coherent monochromatic radiation is incident on a phase optical element in a direction normal to its surface. Denote the diffraction angle by α, the spectral coordinate by $u = \sin(\alpha)/\lambda$, and the wavelength by λ. Then the intensity of light diffracted in the given direction is [54]

$$I(u) = |F(u)|^2 |\sin(\pi N_0 ud)/\sin(\pi ud)|^2, \tag{2.1}$$

where N_0 = number of repetitions (periods), d = period, and $F(u)$ denotes (accurate to a constant factor) the Fourier transform of the phase function of period $p(x)$ [54, 55]:

$$F(u) = \int_0^d \exp(-2\pi iux)\exp(ip(x))\,dx. \tag{2.2}$$

The calculated elements were as a rule implemented as elements of plane optics [30]. Therefore the natural profile was piecewise constant (raster), dividing the period into N elements of size $\delta = d/N$. For this profile Eq. (2.2) reduces to

$$F(u_n) = \text{const}\cdot\text{sinc}\left(\frac{n}{N}\right)\sum_{k=0}^{N-1}\exp(-2\pi ikn/N)\exp(ip_k), \tag{2.3}$$

where $u_n = n/d$ stands for the main directions (maxima) of the grating, and $|u_n| < 1/\lambda$. Equation (2.3) allows the main maxima of the grating field to be calculated directly through the fast Fourier transform (FFT), a simple and efficient way of modeling and solving this problem on a computer.

Equations (2.1) and (2.3) define the field in the entire domain of variation of u. The problem thus boils down to determining phases p_k, which offer the best approximation $|F(u_n)|$, that is, RP to the specified pattern. It suffices to consider $F(u_n)$ only, as $F(u)$ is readily recovered by $F(u_n)$ (sampling theorem).

As a figure of merit of the goodness of fit of the radiation pattern we choose

$$\varepsilon = \|\tilde{a} - \langle\tilde{a}\,|\,\tilde{b}\rangle\tilde{b}\|^2, \tag{2.4}$$

where $\tilde{a} = a/\|a\|$, $\tilde{b} = b/\|b\|$ are the normalized vectors $a = \{a_n\}$, $b = \{b_n\}$, where a_n are the given values $|F(u_n)|$, and b_n are the calculated values. Here

$$\|a\|^2 = \sum a_n^2, \qquad \langle a\,|\,b\rangle = \sum a_n b_n.$$

Thus, ε has the physical meaning of the energy of the compensating field which should be added to the calculated field to obtain precisely the given radiation pattern (relative energy with the given field assumed to be unity).

An iterative algorithm of the Herchberg–Saxton type [82] was chosen as the basic computational algorithm. The associated constraints involved both spatial and frequency (spectral) domains, namely amplitude correction and phase quantization correction for diffraction at a mask.

2.2. *Computer-simulations of directional patterns*

Equalizing the beam within a given angle is an interesting problem because of its numerous applications. By constructing the directional pattern of type 1 within the given angle, and of type 0 outside it, one may, for example, develop a phase diffraction grating that splits a plane wavefront (collimated beam) into a fan of similar plane waves (beams), a multifocus lens with a linear or rectangular array of foci (with identical energy distributions at each focus), or elements that focus coherent radiation into a segment or a rectangle.

Diffraction gratings that separate a collimated beam into a fan of several dozens of similar beams of about equal energies (having a max/min ratio of the energy values within 1.5–2) may be applied in systems of holographic interferometry [5, 6]; thus making feasible the optical tomography of high-speed processes such as plasma discharges, which are only microseconds long. Such computer-generated optical elements with a complicated phase function drastically simplify the optical system of plasma illumination and holographic recording, since the entire information is recorded on one or two holograms with up to 100 pixels of resolution for penetration depth.

Considerable practical interest is attached to gratings with a symmetrical array of diffraction orders and energies equal in all specified orders (in practice the energies are not equal, but proportional to $1/\mathrm{sinc}(n/N)$, considering the diffraction at the mask (raster) array).

Table 1 summarizes errors in calculation for a grating with diffraction orders $-B, \ldots, 0, \ldots, +B$ $(2B+1=K)$ and equal energies in these orders (in other orders the energy is specified to be zero).

A way to solve this kind of problem was first demonstrated by Dammann and Gortler [80]. They handled the problem of a phase diffraction grating with equal energies in orders $-B, \ldots, 0, \ldots, +B$ assuming that the phase function of the grating period was binary-quantized, but without discretization. Their formulation was associated with the available technology of photomask fabrication and photolithographic etching. They had to solve a nonlinear system of equations of dimensionality B under the assumption of a symmetric phase function of the period (otherwise this dimension would be $2B+1$, and size reduction would be a major problem). They managed to obtain solutions for B up to 8. Turkevich and Bobrov [62] extended their results to $B=11$; and failed to find a solution for higher B, though it most probably exists.

As we formulated the problem with discretization of phase function and quantization at an arbitrary number of levels, dimensionality problems are practically absent, though convergence is slower, especially for close solutions. In such a case if a more accurate levelling of energies in the orders $-B, \ldots, 0, \ldots, +B$ is required the algorithm should be modified for the final steps. Outside this region unwanted diffraction orders are always well suppressed—almost 98–99% of the total energy goes into the orders $-B, \ldots, 0, \ldots, +B$. The magnitude of B determines the angle within which the directional pattern is equalized. For an element with N elements of the array, B should be less than N.

There exist, however, two factors that distort the real direction pattern which is to be made rectangular. The first of them is analogous to the Gibbs effect. If a rectangular radiation pattern is desirable, i.e. a pattern sharply out at the edges, then amplitude oscillations appear inside the working zone and immediately beyond the edge. The magnitude of these oscillations cannot be reduced by computational means.

This effect may be suppressed to a considerable degree if the directional pattern is specified not as rectangular, but, for instance, as a trapezoid, that is, with smoothly cut edges. The mean square deviation of this directional pattern is higher from the rectangular than from the oscillatory pattern; however, in the portion where the pattern amplitude is given as constant, the accuracy is greater.

Table 1. Values of ε for optimal gratings as a function of the number of orders K (with equal energies) and the number of elements of the discretization array

	K						
N	5	7	9	15	31	37	73
8	0.017	0.00008					
16	0.090	0.027	0.070	0.002			
32	0.110	0.026	0.061	0.034	0.001		
64	0.109	0.025	0.056	0.040	0.030	0.014	
128	0.109	0.025	0.055	0.039	0.037	0.033	0.018

This implies that a good (down to 1–2%) equalization within the working zone of the beam is achieved at the expense of a certain loss of energy (usually under 5%).

The second distorting factor is the fall of diffraction efficiency with spatial frequency caused by the directional pattern of each element of the array. For an exact step-profile, this factor is considered by the multiplier $\mathrm{sinc}(n/N)$, but for actual elements the array profile is never exactly rectangular and the real reduction of diffraction efficiency is somewhat higher.

This effect has a useful side, namely, one may use the variation of relief profile on an array element, made to ensure a steeper fall of the experimental curve, effectively to suppress the sidelobes, keeping the programmed correction of the pattern in the central zone (main lobe).

Elements with asymmetric or complicated directional pattern. Computer analyses have indicated that given any distribution of a_n (i.e. directional pattern), phases p_k can be found that ensure it with an acceptable accuracy. This accuracy is hard to predict in advance, but, as a rule, it increases for increasing numbers of non-zero a_n.

Two-dimensional elements. For a factorizable directional pattern $F(u, v) = A(u)B(v)$, where u and v are the spectral coordinates corresponding to x and y, the phase functions providing the desired directional pattern can be sought independently in x and y. These phase functions are combined to give the desired two-dimensional phase function (Fig. 2).

Computer simulation experiments indicate that, even in this simplest version, the two-dimensional case imposes stricter requirements on both accuracy of the solutions and precision element manufacture. The main reason is a coherent cross-interference of the diffracting wavefronts due to inaccuracies of the calculations and imprecision of manufacture in each of the coordinates and between them.

For the IR (10.6 μm) range, elements with the square, net-array, and quasilinear directional patterns were prepared. The geometrical dimensions of the patterns were maintained with a

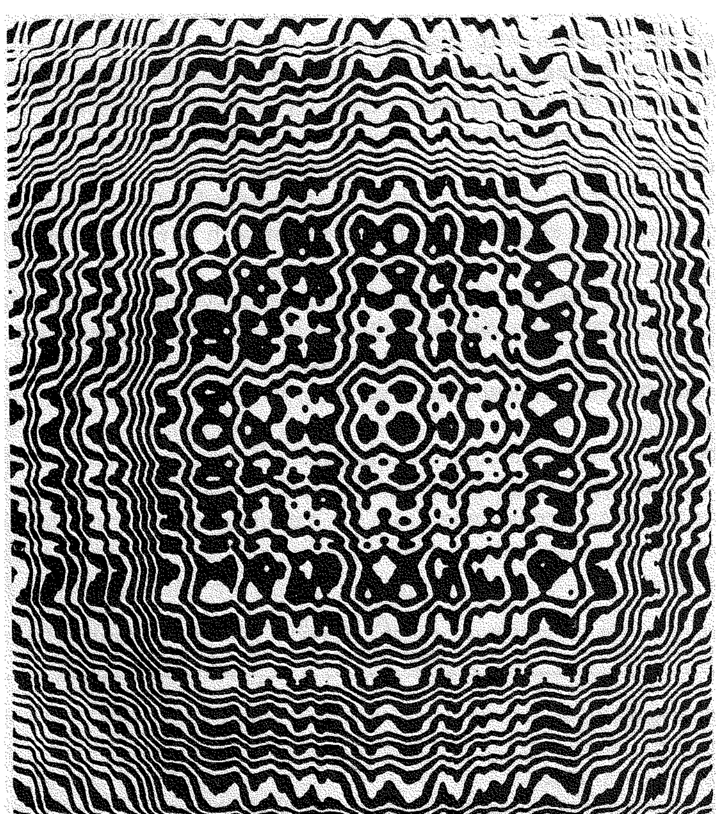

Fig. 2. Factorizable two-dimensional phase function for focusing into a square (with an additional lens phase function).

sufficiently high accuracy (suppression where $a_n = 0$) sizeable intensity fluctuations inside the patterns were observed due to the aforementioned causes.

2.3. *Effect of errors in phase specification on directional pattern*

Errors due to imperfect technology of element manufacture influence the directional pattern of a phase optical element most of all. The general methodological errors typical of all technologies are clipping (multiplication by a factor close to unity) of phase and nonlinearity of its recording. Denote by f the phase, $0 \leqslant f < 1$, then for the element transmittance function we have $f \to r(f)$

$$e^{ir(f)} = \sum_{n=-\infty}^{+\infty} c_n e^{inf}; \qquad \sum_n |c_n|^2 = 1.$$

When the phase transmission is perfect (perfect technology), $c_1 = 1$, and $c_n = 0$ for all other n. In real situations, $|c_n|^2$ is the energy of additional wavefronts with a multiple phase. For elements with a carrier (holograms), this is the energy escaping the intended purpose of the diffractive system, and for elements with a lens (see below) this is the energy directed to additional (real and imaginary) foci equal to F/n, where F is the focal distance of the added lens. In the second case, these energies can be measured directly and used to determine the real phase transfer function. This allows corresponding corrections to be made in the calculations. The quantity $1 - |c_1|^2$ represents the actual losses due to nonlinearity.

The best way to study clipping (uncertainty in relief depth control) is with practical examples. For an element with a small intrinsic error of directional pattern, the error caused by clipping changes as shown below.

$1+C$	0.7	0.8	0.9	0.95	0.98	1.00	1.01	1.05	1.1
ε	0.23	0.11	0.03	0.008	0.001	0.000	0.000	0.01	0.03

The raster array is responsible for the sidelobes

$$F(u_n) = \operatorname{sinc}\left(\frac{n}{N}\right) \sum_{k=0}^{N-1} e^{2\pi ikn/N} e^{iP_k}.$$

The periodic (in n) function under the summation symbol is modulated by multiplying by $\operatorname{sinc}(n/N)$. Actually, N amplitudes of $F(u_n)$ are calculated. Their physical number is about $2d/\lambda$ (λ is the wavelength). Therefore, only the central portion (halfwidth) of the main lobe is calculated (zones such that $|n| \leqslant N/2$). To suppress radiation outside the given angle, it would be wise to use $B \leqslant N/4$, for then bursts of energy can leak only in sidelobes. Since $N\delta = d$, the number of sidelobes is a function of the ratio of the size of an array element, δ, to the wavelength. When the array element size is less than half the wavelength, no sidelobes (in this particular sense) occur.

The motivation for studying the effect of phase quantization on element quality arises primarily from the technology of element manufacture. Photolithography seems to be a widely accepted method of manufacture of optical element. This process is already well known in the integrated circuit manufacture for high accuracy and availability. Its associated technological equipment allows for multistage etching with the use of several photomasks. Calculations indicate that for all gratings at $M = 32$, phase quantization errors become negligibly small, and at M equal to 16 or 8 they, as a rule, are fairly tolerable. Two examples are summarized in Tables 2 and 3.

Table 2. ε for gratings with 15 orders of equal energy at $N = 32$ and $N = 64$ as a function of the number M of phase quantization levels

	M					
N	2	4	8	16	32	∞
32	0.20	0.15	0.07	0.04	0.03	0.03
64	0.18	0.19	0.08	0.05	0.04	0.04

Table 3. ε for gratings with 31 orders of equal energy at $N = 32$ and $N = 64$ as a function of the number M of phase quantization levels

	M					
N	2	4	8	16	32	∞
32	0.11	0.07	0.03	0.001	0.001	0.001
64	0.35	0.08	0.07	0.04	0.03	0.03

2.4. *Experimental studies of elements synthesized for the* 10.6-μm *IR range*

For experimental investigation of directional patterns of array-type phase-optical elements, we selected a technology in which a relief is formed on gelatine followed by vacuum deposition of thin aluminium and copper films [30]. Photomasks (multilevel masks for gelatine hardening, Fig. 3) were manufactured on the raster scanning generator for photographic film R-1700. The elements were exposed to a 10.6-μm, multimode, coherent, illuminating beam. Since the phase function of a lens of 400-mm focal length was added to the phase function of the element, Fraunhofer diffraction was observed in the focal plane of this lens. It had the form of an array of dots (main maxima) for a grating, or the form of a rectangle, straight line section, etc. for an aperiodic element (kinoform).

The energies of the main maxima were measured by a calorimeter. In addition, the heating effect of the irradiation on paper, acrylic plastic and other materials was observed. The accuracy of calorimetric measurements was 8–10% of the magnitude of each measured quality. Measurements were also made of the total energy of the incident irradiation, the energy in the main maxima of the grating, and in the auxiliary (multiple) focal planes that occur owing to the nonlinearity in recording the phase. As a rule, losses due to diffuse scattering, sidelobes and substantial nonlinearity of phase recording reduce the total energy $|c_1|^2$ in the main focal plane to 40–50% of the beam energy. The values of $|c_2|^2$ about 15%, $|c_0|^2$ about 10%, etc., imply that the nonlinearity was

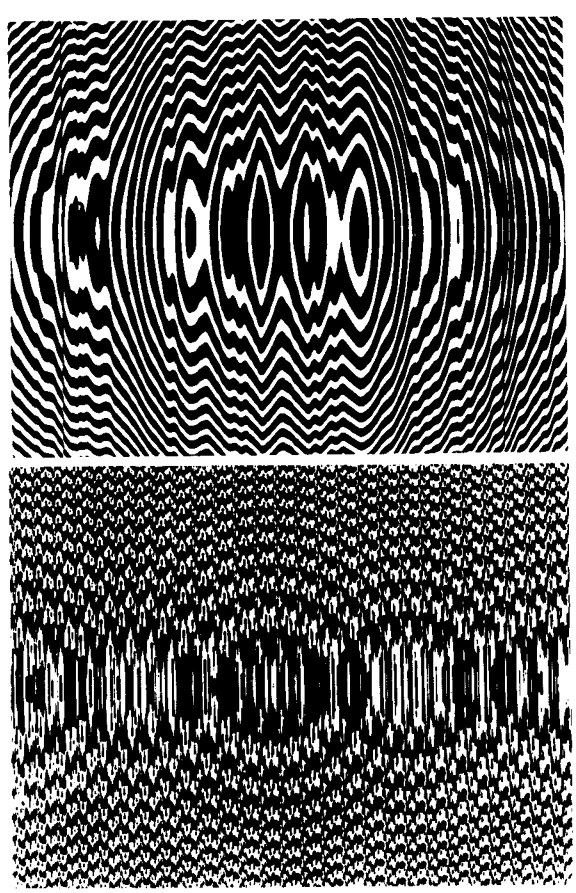

Fig. 3. Photomasks for phase diffraction gratings (with an additional lens).

substantial. The accuracy of measurements was insufficient to correct the phase transfer function with acceptable accuracy. This function was unstable because of instability of photomaterials and processing procedure used. Measurements with an interference microscope estimated the magnitude of clipping (uncertainty in relief depth control) to be about 10% but it was not quite uniform over an element's area.

Calorimetric studies indicated that the directional pattern in the main focal plane practically coincided with the given shape, since here it was influenced mainly by slipping. Clipping-induced distortion of directional patterns was as low as $\varepsilon = 0.03$, that is, it could not be detected by visual observations or with the aid of a calorimeter.

To measure how diffraction efficiency decreases with spatial frequency on a real mask array, a grating with $N = 32$ array elements was manufactured, and $a_n = 1$ at $|n| \leqslant 6$ was specified. The usual correction of a_n to allow for this reduction of diffraction efficiency was not carried out.

n	0	1	2	3	4	5	6
Theory	1	0.99	0.95	0.89	0.81	0.71	0.61
Experiment	1	0.96	0.87	0.77	0.66	0.62	0.57

The evidence collected in this experiment confirmed the assumption that the technological smoothing of array profile substantially affects the rate of decrease of diffraction efficiency with spatial frequency and also sidelobe size.

This method offers a rapid way to control the directional pattern of array phase optical elements with limited computational resources. The time needed is so short that the method may be used to tackle real-time problems, for example, with the use of reversible holographic media.

The outlined method of calculating phase diffraction gratings with specified parameters allows a number of problems to be solved over the control of electromagnetic radiation in the visual, IR and other ranges. The relative simplicity of the calculations and ease of manufacture of these gratings commend this method in a wide range of applications in physics and technology.

3. SYNTHESIS AND STUDIES OF PHASE SPATIAL FILTERS FOR SPECIFIED CURVILINEAR COORDINATE TRANSFORMATIONS

An operation typical of optical data processing (with coherent and incoherent beams) is the transformation of coordinates in the image, that is, the transformation of a light field in which to a point in the input plane of the system there corresponds a point in the output plane. Here, the correspondence is understood in the sense of geometrical optics.

Problem formulations on obtaining a desirable transformation with the help of computer-generated holograms have been reiterated in the literature (see, e.g. [18, 71–75]). In effect, to obtain a desired phase function one has to solve a system of first-order partial differential equations.

This chapter considers when this system of equations is integrable, i.e. when the desired phase hologram can be obtained in principle, and solution techniques (integration) for this system with the use of complex functions. One of the promising applications—holographic pattern recognition (by two-dimensional correlation) will also be covered. This method is invariant to rotation, changes of scale and shifts in the input signal, rather than to shift only as is the case with conventional holographic correlators.

3.1. *Phase filters for curvilinear coordinate transformations*

Consider the system of coordinate transformation using the frequency plane of a lens [71]. If we denote the (dimensionless) phase function of an optical element by $F(x, y)$, then in the paraxial region we have

$$u(x, y) = \frac{f}{K} F_x(x, y),$$
$$v(x, y) = \frac{f}{K} F_y(x, y), \tag{3.1}$$

where

(x, y) = Cartesian coordinates of a ray in the front focal plane of the lens in which the element is placed;

(u, v) = Cartesian coordinates of the ray in the back focal plane of the lens in which the transformed image is formed;

f = focal length of the lens;

k = wavenumber;

F_x, F_y = partial derivatives of $F(x, y)$ with respect to x and y.

We shall treat (3.1) as the system of equations for solving the inverse problem formulated as follows: find the phase function $F(x, y)$ of an element that would perform the specified transformation for the given functions $u(x, y)$ and $v(x, y)$.

Clearly, a necessary condition for $F(x, y)$ to be smooth is that $u_y = v_x$. It can be demonstrated [32, 59, 63] that this condition is sufficient for single connected domains. This condition actually means that the integral along the curve $u\,dx + v\,dy$ is independent of the path of integration. This affords an easy construction of the functions $u(x, y)$ and $v(x, y)$ by numerical integration of the respective integrals on a computer.

To obtain an optical system for the coordinate transformation without resorting to a lens one may use directly the equations of geometrical optics [36]. The initial system of equations is then

$$\begin{aligned} u &= x + sP_x(x, y), \\ v &= y + sP_y(x, y) \\ s &= f/(1 - P_x^2 - P_y^2)^{1/2}, \end{aligned} \tag{3.2}$$

where

f = distance between the plane of the optical element and the observation plane;

s = optical path length of a ray;

P is the eikonal: $F(x, y) = kP(x, y)$.

Having transformed this system to a form similar to (3.1), we get:

$$\begin{aligned} P_x &= X/(1 + X^2 + Y^2)^{1/2}, \\ P_y &= Y/(1 + X^2 + Y^2)^{1/2}, \end{aligned} \tag{3.3}$$

where $X = (u - x)/f$ and $Y = (v - y)/f$.

In the paraxial approximation

$$\begin{aligned} P_x &= X, \\ P_y &= Y. \end{aligned} \tag{3.4}$$

As might have been expected the integrability condition $X_y = Y_x$ coincides with that of the existence of a paraxial region.

In many cases of practical significance the solution of the system, i.e. the phase function $F(x, y)$, can be obtained explicitly without numerical integration with the associated shortcomings such as complicated calculations, need for the errors to be taken into account, provision of stability, and the like. This possibility stems from the observation that the condition of existence coincides with one of the two Cauchy–Riemann equations for complex antianalytical functions [43], that is, functions of the type $f(z^*)$, where $f(z)$ is a complex analytical function (say a polynomial, sin, ln or exp), and the asterisk denotes complex conjugation.

Thus, if transformation $(x, y) \to (u, v)$ written in the complex form, $w = f(z^*)$, $w = u + iv$, $z^* = x - iy$, is an antianalytical function, then the condition of existence is satisfied, and also the second Cauchy–Riemann equation. Integration can then be written as

$$u\,dx + v\,dy = \mathrm{Re}\{f(z^*)\,dz^*\},$$

where $dz^* = dx + i\,dy$.

This leads to an explicit formula for $F(x, y)$,

$$F(x, y) = \mathrm{Re}\left\{\int f(z^*)\,dz^*\right\}. \tag{3.5}$$

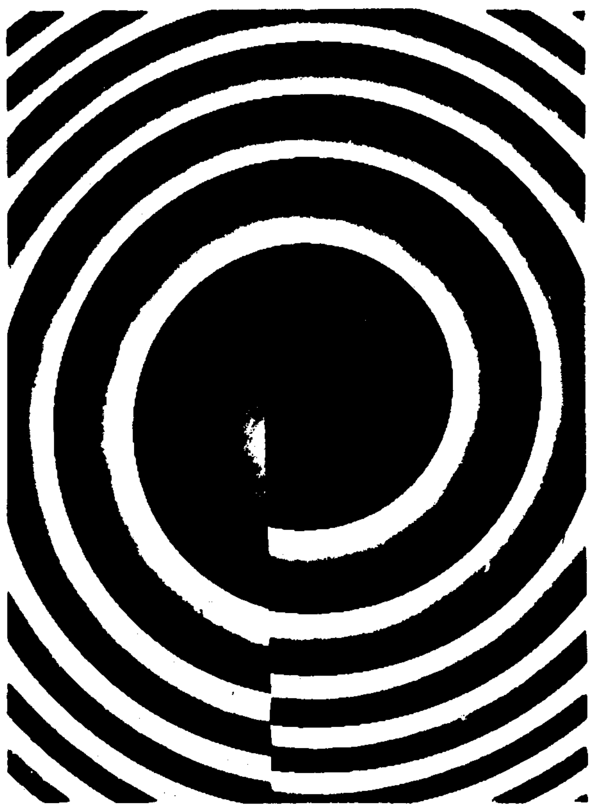

Fig. 4. Phase function (3.7) with additional lens.

Thus, the transformation

$$(x, y) \to (\theta, \ln r) \tag{3.6}$$

is feasible, and the corresponding phase function

$$F(x, y) = x\theta + y \ln r - y. \tag{3.7}$$

The corresponding photomasks are shown in Figs 4 and 5.

3.2. *Invariant pattern recognition based on coordinate transformation*

One of the main problems for holographic systems of pattern recognition and other optical systems based on spatial filtering is the strong dependence of the spatial filter on distortions of the input (two-dimensional) signal. In systems of holographic pattern recognition these distortions are essentially changes of scale and rotations of the input signal (i.e. the pattern to be recognized). For more detail see e.g. [74]. Rotation of an input pattern through 2–3° with respect to the spatial filter, or a change of scale by 2–3% render the system completely inoperable.

Many workers have attempted to overcome these difficulties (at least partially). One approach [74, 92] seems to be more logical than others. It suggests a pattern-recognition system that transforms input signal distortions (whose type is known in advance, but not the parametric magnitudes) in the final analysis into shifts, thus rendering the system invariant to this type of distortions. It will be able to recognize even a severely distorted pattern, provided that distortions are thus transformed. Moreover, the magnitude of a distortion can be determined by the magnitude of the shift as it leads to a corresponding displacement of the correlation peak at the system output.

Thus, the coordinate transformation (3.6) is necessary for pattern recognition to be invariant to shifts, rotations and changes in scale. Casasent and Psaltis [74] assumed that this transformation would have to be done in two steps. One was to be made on a computer (i.e. a hybrid processor had to be built) and the other was to be implemented with the aid of an artificial hologram (Mellin transform) or a special-pattern input system with logarithmic deviation of the electron beam.

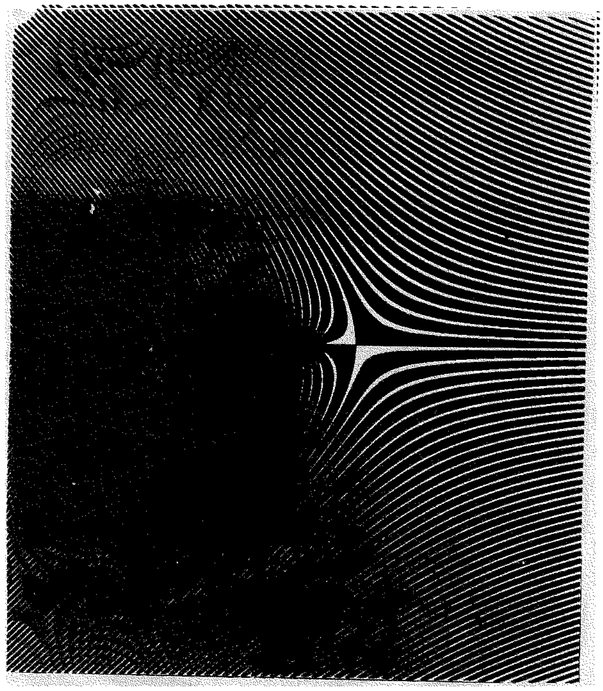

Fig. 5. Phase function (3.7).

The results outlined above indicate, however, that this transformation can be realized in one step with a single synthesized hologram with phase function (37). For this purpose, certain coefficients should be introduced into (3.7), to allow for that specific realization. These are f, k, the scale M in the output plane and the average radius r_0 of the input signal. The modified formula will then be

$$F(x, y) = \frac{Mk}{f}\left[x\theta + y\ln\left(\frac{r}{r_0}\right) - y\right]. \tag{3.8}$$

The opportunity to perform such optical transformations as a change of image coordinates with the aid of generated holograms or phase elements affords new technologies for building flexible, reliable and powerful optical systems for technical vision based on two-dimensional Fourier transforms.

This opens up new fields for optical data processing.

3.3. *Physical limitations due to discretization, quantization and nonparaxiality*

As shown in Section 1, irrespective of phase filter structure, discretization of the signal imposes limitations on its maximum spatial frequencies (for images, the limit is the spatial resolution $1/2d$ lines/mm; for the device under study, the resolution was 20 lines/mm), and results in some loss of energy due to diffraction in the array that does not overlap with the output signal.

Similarly, the effect of quantization and nonlinear distortions of the phase function should be considered separately for plane optics elements and for generated phase holograms. All conclusions of Section 1.2 are valid here as well.

3.4. *Experimental studies of computer generated filters*

For synthesis of coordinate-transformation elements with the aid of a pattern generator a program was written analogous to that for generation of Bessel optics elements. We synthesized a test element with the following parameters:

— frequency of the carrier close to the maximum (about 15 lines/mm);

— image dimensions, controlled by the scaling coefficient M and the focal length of the Fourier-lens, f, were $l_x = 2.8$ mm, $l_y = 8.6$ mm with the aperture $r_{max} = 12.8$ mm, $f = 750$ mm (0.2 lens), and function (3.8).

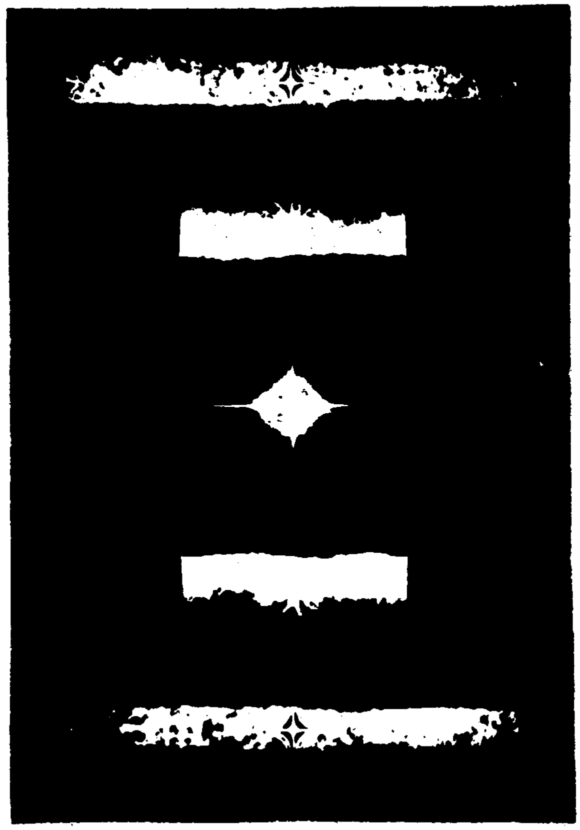

Fig. 6. Transform (3.6) of a circular uniformly illuminated aperture taken with the aid of a hologram (3.7).

The Fourier system was assembled on the basis of the UIG-2M set with an LG-52 He–Ne laser.

The transform of the circular uniformly illuminated aperture is shown in Fig. 6.

The following deviations of the resulting transformation from specification were observed. When the aperture diaphragm was displaced with respect to the centre of the element, a straight line section $r = \text{const.}$ acquired an S-shaped bend with $l_y b_r / r_{\max}$ proportional to the displacement b_r. In the transformed image the line width was close to that estimated by the formula

$$D_1 D_2 = \text{const.} \times \lambda f,$$

where

D_1 = initial line width, in the initial pattern;

D_2 = line width in the output image;

λ = wavelength (0.633 μm), $f = 750$ mm.

The image of the operation edge also deviated from the rectilinear owing to distortion and bending of the photomask (filter on Kodak-649 film), phase noise of the substrate and emulsion (immersion was not used), nonzero curvature of the wavefront on the element caused by inaccuracy of system alignment, and also because the paraxial approximation was involved.

The transformation calculated in the paraxial approximation by Eq. (3.6) with the phase function (3.7) will differ from the given transformation (3.6). As follows from (3.1) the difference will manifest itself in the form of a factor, close to unity,

$$(1 - P_x^2 - P_y^2)^{-1/2}, \qquad |P_x| \ll 1, \quad |P_y| \ll 1, \tag{3.9}$$

that appears at the functions $u(x, y)$ and $v(x, y)$.

From Eqs (3.5), $P_x \cong X$, $P_y \cong Y$ accurate to the terms of the 3rd-order of smallness in $|X|$ and $|Y|$. It follows that the leading term of the expansion of (3.9) into a Taylor series has the form:

$$((u - x)^2 + (v - y)^2)/(2f^2).$$

This allows us to estimate the deviation from the exact transform

$$\tilde{u} = u(1 + e),$$
$$\tilde{v} = v(1 + e),$$
$$e \cong (x^2 + y^2)/2,$$

where $X = (u - x)/f$ and $Y = (v - y)/f$ are the angles that the rays make with the axes x and y.

For example, for $|X|, |Y| < 1/10$ we have $e < 1/100$. In our experiments $f = 750$ mm; $|x|, |y| < 13$ mm, $|u| < 3$ mm, $|v| < 5$ mm yield the estimate $e < 1/200$, which implies that the distortion was negligibly small.

REFERENCES

1. M. Adams. *Introduction into the Theory of Optical Waveguides*. Mir, Moscow (1984) (Russian translation).
2. A. Azarov, I. Bagbaya, A. Bereznyi, I. Sissakian and V. Soifer. Phase optical element with a specified directional pattern. *Komputernaya Optika* No. 1, 116–127 (1987).
3. A. Andreev, A. Bereznyi and I. Sissakian. Phase optical elements for analytic coordinate transformations. *Komputernaya Optika* No. 1, 151–153 (1987).
4. Yu. A. Ananiev. *Opticheskie resonatory i problema raskhodimosti lasernogo izlucheniya* [*Optical Resonators and Divergence of Laser Beam*]. Nauka, Moscow (1979).
5. I. Bagbaya, A. Bereznyi, I. Sissakian, V. Soifer and A. Shwartzburg. Reconstructive tomography of thermonuclear plasma by digital holography. In: *Opticheskaya zapis i obrabotka informatsii* [*Optical Data Recording and Processing*], pp. 43–50. KUAI Press, Kuibyshev (1986).
6. I. Bagbaya, I. Bereznyi, I. Sissakian, V. Soifer and A. Shwartzburg. A method of holographic interferometric visualization of nonuniformities in 3D phase objects. USSR author invention certificate 1340291, 28.06.86 (in Russian).
7. L. Bakhrakh and S. Kremenetsky. *Sintez isluchayushchikh sistem: teoria i metody rascheta* [*Synthesis of Irradiating Systems: Theory and Design*]. Sovietskoye Radio, Moscow (1974).
8. L. Bakhrakh and A. Kurochkin. *Golografia v mikrovolnovoi tekhnike* [*Holography in Microwave Engineering*]. Sovietskoye Radio, Moscow (1979).
9. L. Bakhrakh and O. Litvinov. Dependence of amplitude and phase of wavefields. *Dokl. Akad. Nauk USSR* **266** (2), 234–237 (1982).
10. A. Bereznyi, L. Brusilovsky, E. Otlivanchik, D. Sagatelyan and I. Sissakian. Draft computerized system for design synthesis, analysis and application of elements of plane optics. *Komputernaya Optika* No. 2, 21–29 (1987).
11. A. Bereznyi, A. Prokhorov, I. Sissakian and V. Soifer. Bessel-optics. *Dokl. Akad. Nauk USSR* **274** (4) (1984).
12. A. Bereznyi, A. Prokhorov, I. Sissakian and N. Soifer. Bessel transformation in optical data processing. *5th All-Union School in Optical Data Processing*. Kiev (1984) (in Russian).
13. A. Bereznyi and I. Sissakian. Optical transformations of coordinates. In: *Opticheskaya zapis i obrabotka informatsii* [*Optical Data Recording and Processing*], pp. 22–28. KUAI Press, Kuibyshev (1986).
14. A. Bereznyi and I. Sissakian. Optical systems for image coordinate transformation based on generated phase optical elements. *2nd All-Union Conference on Optical Image Formation*. Kishinev (1985) (in Russian).
15. A. Bereznyi and I. Sissakian. Binary elements of Bessel-optics. *Komputernaya Optika* No. 1 (1987).
16. A. Bereznyi, S. Komarov, A. Prokhorov, I. Sissakian and V. Soifer. Phase diffraction gratings with given parameters. *Dokl. Akad. Nauk USSR* **287** (3), 632–627 (1986).
17. A. Bereznyi, A. Prokhorov, I. Sissakian and V. Soifer. Optical element for Fourier–Bessel transform of given order. Author certificate by application No. 4935339/10, 13.12.85 accepted 01.12.86 (in Russian).
18. O. Bryngdahl. Optical transformations. *Avtometria* **2**, 30 (1983).
19. T. Huang (Ed.). *Fast Algorithms in Digital Image Processing*. Radio i Svyaz, Moscow (1984) (Russian translation).
20. G. Vasilenko and L. Tsibulkin. *Gologrficheskie raspoznayushchie ustroistva* [*Holographic Pattern Recognition*]. Radio i Svyaz, Moscow (1985).
21. G. Vasilenko. *Teoria vosstanovlenia signalov* [*Theory of Signal Reconstruction*]. Nauka, Moscow (1979).
22. A. Vasin, M. Golub, V. Danilov, N. Kazantsky, I. Sissakian, V. Soifer and G. Uvarov. Design and analysis of coherent wavefield in the focal spot of a rotationally-symmetric optical element. *Reprint FIAN* No. 304 (1983) (in Russian).
23. *Vichislitelnaya optika* [*Computational Optics*]. Mashinostroyenie, Leningrad (1984).
24. M. Golub, S. Krivoshlykov, A. Prokhorov, I. Sissakian and V. Soifer. Spatial filters for analysis and synthesis of a transverse-mode structure of coherent electromagnetic radiation. *Reprint FIAN* No. 21 (1983) (in Russian).
25. J. W. Goodman. *Introduction to Fourier Optics*. McGraw-Hill, New York (1982).
26. O. Svelto. *Principles of Lasers*. Plenum Press, New York (1982).
27. E. Zelkin and R. Petrova. *Linzovye antenny* [*Lens Antennas*]. Sovietskoye Radio, Moscow (1974).
28. E. Zelkin and V. Sokolov. *Metody sinteza antenn* [*Methods of Antenna Synthesis*]. Sovietskoye Radio, Moscow (1980).
29. V. Ivanchenko and G. Poskonny. Spectral analysis of signals in optoelectronic systems with spatially noncoherent radiation source. *Avtometria* No. 2, 52–56 (1983).
30. V. Danoliv, V. Popov, A. Prokhorov, D. Sagatelyan, I. Sissakian and V. Soifer. Optical element focusing coherent radiation along arbitrary focal line. *Preprint FIAN* No. 69 (1983) (in Russian).
31. A. Gorelik, I. Gurevich and V. Skripkin. *Sovremennoe sostoyanie problemy raspoznovaniya* [*State of the Art in a Pattern Recognition Problem*]. Radio i Svyaz, Moscow (1985).
32. E. Kanasevich. *Analiz vremennykh posledovatelnostei v geofizike* [*Analysis of Time Sequences in Geophysics*]. Nauka, Moscow (1985).

33. I. S. Klimenko. *Golografia sfokusirovannykh izobrazhenii i spektrointerferometria [Holography of Focused Images and Spectrointerferometry]*. Nauka, Moscow (1977).
34. B. Frieden (Ed.). *The Computer in Optical Research*. Springer, Berlin (1980).
35. V. Koloborodov. Errors in assembling and aligning of coherent optical spectroanalyzers. *Opt. Mekh. Promst.* No. 9, 6 (1983).
36. Yu. Kravtsov and Yu. Orlov. *Geometrical Optics of Inhomogeneous Media*. Springer, Berlin (1989).
37. R. Kyun. *Mikrovolnovye antenny [Microwave Antennae]*. Sudostroyanie, Leningrad (1967).
38. F. Laeri and T. Tigudi. Coherent optical feedback. *Avtometria* No. 2, 15–17 (1983).
39. M. Lavrentiev, K. Reznitskaya and V. Takhno. *Odnomernye obratnye zadachi matematicheskoi fiziki [One-dimensional Inverse Problems of Mathematical Physics]*. Nauka, Novosibirsk (1982).
40. S. A. Mayorov, E. F. Ochin and Yu. F. Romanov. *Opticheskie analogovye vychislitelnye mashiny [Optical Analog Computers]*. Energoatomizdat, Leningrad (1983).
41. D. I. Mirovitsky, I. F. Budagian and V. F. Dubrovin. *Mikrovolnovaya tekhnika i golografiya [Microwave Engineering and Holography]*. Nauka, Moscow (1983).
42. A. Maitland and M. H. Dunn. *Laser Physics*. North-Holland, Amsterdam (1969).
43. A. F. Nikiforov and V. B. Uvarov. *Specialnye funktsii matematicheskoi fiziki [Special Functions of Mathematical Physics]*. Nauka, Moscow (1978).
44. I. D. Nikolov. Mirror systems in data processing systems. *Avtometria* No. 6, 58 (1984).
45. M. Bolts (Ed.). *Inverse Problems in Optics*. Mashinostroenie, Moscow (1984) (Russian translation).
46. A. V. Oppenheim and J. S. Lim. Importance of phase in signals. *Proc. IEEE* **69** (5), 529–541 (1981).
47. H. J. Caulfield (Ed.). *Handbook of Optical Holography*. Academic Press, New York (1979).
48. S. B. Gurevich (Ed.). *Optical and Optoelectronic Methods of Image and Signal Processing*. LIYaF Press, Leningrad (1982) (in Russian).
49. I. I. Pakhomov and A. B. Tsibuliya. *Raschet opticheskikh sistem lasernykh priborov [Design of Optical Systems of Laser Equipment]*. Radio i Svyaz, Moscow (1986).
50. Yu. Ye. Polsky. Optical resonators of power gas-discharge lasers. In: *Itogi nauki i tekhniki*. VINITI (1980). *Ser. Radiotekhnika* **21**, Moscow (1980) (in Russian).
51. *Methods of Optical Data Processing and Holography*. LIYaF, Leningrad (1980) (in Russian).
52. *Direct and Inverse Problems of Diffraction Theory*. IRE AN USSR Press, Moscow (1979) (in Russian).
53. V. G. Romanov. *Obratnye zadachi matematicheskoi fiziki [Inverse Problems of Mathematical Physics]*. Nauka, Moscow (1984).
54. M. Rousseau and J. P. Mathieu. *Problems in Optics*. Pergamon Press, Oxford (1973).
55. D. Sivukhin. *Optika [Optics]*. Nauka, Moscow (1980).
56. V. A. Soifer. *Tsifravaya golografia i eyo primenenie [Digital Holography and its Applications]*. KuAI Press, Kuibyshev (1978).
57. L. M. Soroko. *Gilbert-optica [Hilbert Optics]*. Nauka, Moscow (1981).
58. L. M. Soroko. *Osnovy golografii i kogerentnoi optiki [Fundamentals of Holography and Coherent Optics]*. Nauka, Moscow (1971)
59. A. N. Tikhonov and V. Ya. Arsenin. *Metody resheniya nekorrektnykh zadach [Solution Techniques for Ill-posed Problems]*. Nauka, Moscow (1986).
60. B. J. Thompson. Hybrid processing systems—an assessment. *Proc. IEEE* **65** (1), 62–76 (1977).
61. A. M. Trakhman. *Vvedenie v obobshchennuyu spektralnuyu teoriyu signalov [Introduction to Generalized Spectral Theory of Signals]*. Nauka, Moscow (1972).
62. Yu. Turkevich, S. Bobrov and G. Greisukh. *Optika diffraktsionnykh elementov i sistem [Optics of Diffraction Elements and Systems]*. Mashinostroenie, Leningrad (1986).
63. I. Hirschman and G. Widder. *The Convolution Transform*. Princeton, NY (1955).
64. R. Hunsferger. *Integrated Optics: Theory and Technology*. Springer, Heidelberg (1982).
65. L. P. Yaroslavsky and N. S. Merzljakov. *Metody tsifrovoi golografii [Methods of Digital Holography]*. Nauka, Moscow.
66. L. P. Yaroskavsky and N. S. Merzljakov. *Tsifrovaja golografiya [Digital Holography]*. Nauka, Moscow (1982).
67. V. A. Zverev and N. S. Stepanov (Eds). *Eksperimentalnaya radiooptika [Experimental Radiooptics]*. Nauka, Moscow (1977).
68. R. A. Athale, H. H. Szu and J. N. Lee. Optical implementation of integral transforms with Bessel kernels. *Opt. Lett.* **7** (3), 124–126.
69. J. J. Bennett. Achromatic combinations of hologram optical elements. *Appl. Opt.* **15** (2), 542–545 (1976).
70. K. Biedermann and O. Holmgren. Large size distortion-free computer-generated holograms in photoresist. *Appl. Opt.* **16** (8), 2014–2016 (1977).
71. O. Bryngdahl. Optical map transformations. *Opt. Commun.* **10**, 164 (1974).
72. O. Bryngdahl. Optical scanner-light deflection using computer-generated diffractive elements. *Opt. Commun.* **15**, 273 (1975).
73. O. Bryngdahl. Computer-generated holograms as generalized optical components. *Optical Engng* **14** (5), 426 (1975).
74. D. Cassasent and D. Psaltis. Deformation invariant optical processors using coordinate transformations. *Appl. Opt.* **16** (8), 2288 (1977).
75. J. Cederquist and A. M. Tai. Computer-generated holograms for geometric transformations. *Appl. Opt.* **23** (18), 3099 (1984).
76. J. J. Clair and C. I. Abitol. Recent advances in phase profiles generation. In: *Progress in Optics*, Vol. 16 (edited by E. Wolf), p. 73. North-Holland, Amsterdam (1978).
77. J. C. Dainty and M. Nieto-Vesperinas. Testing for uniqueness of phase recovery in two dimensions. *Opt. Commun.* **52** (2), 94 (1984).
78. W. J. Dallas. Phase quantization—a compact derivation. *Appl. Opt.* **10** (3), 673 (1971).
79. W. J. Dallas and A. W. Lohmann. Quantization and other nonlinear distortions of the hologram transmittance. *Opt. Commun.* **5** (2), 78–81 (1972).
80. H. Dammann and K. Gortler. High efficiency in-line multiple imaging by means of multiple phase holograms. *Opt. Commun.* **3** (5), 312 (1971).
81. R. Petit (Ed.). *Electromagnetic Theory of Gratings*. Springer, Berlin (1980).

82. J. R. Fienup. Phase retrieval algorithms: a comparison. *Appl. Opt.* **21** (15), 2758–2769 (1982).
83. C. L. Giles and H. H. Szu. Optical implementation of coordinate transformations for pattern recognition. *J. Opt. Soc. Am.* **73**, 1860 (1983).
84. F. O. Huck and S. K. Park. Optical-mechanical line-scan imaging processing: its information capacity and efficiency. *Appl. Opt.* **14** (10), 2508–2520 (1975).
85. W. H. Lee. Computer-generated holograms: techniques and applications. In: *Progress in Optics*, Vol. 16 (edited by E. Wolf), p. 121. North-Holland, Amsterdam (1978).
86. B. Lie and N. C. Gallagher. Convergence of a spectrum shaping algorithm. *Appl. Opt.* **13**, 2470–2471 (1974).
87. A. W. Lohmann and W. T. Rhodes. Two-pupil synthesis of optical transfer function. *Appl. Opt.* **17**, 1141 (1978).
88. A. W. Lohmann and N. Streibl. Map transformations by optical anamorphic processing. *Appl. Opt.* **22**, 780 (1983).
89. R. J. Marks. Coherent optical extrapolation of 2-D bandlimited signals: processor theory. *Appl. Opt.* **19**, 1670–1762 (1980).
90. D. Maystra. Rigorous vector theories of diffraction gratings. In: *Progress in Optics*, Vol. 21 (edited by E. Wolf), pp. 3–70. North-Holland, Amsterdam (1984).
91. D. Psaltis. Incoherent electro-optic image correlator. *Optical Engng* No. 1 (1984).
92. Y. Saito. Scale and rotation invariant real time optical correlator using computer-generated hologram. *Opt. Commun.* **47**, 8 (1983).
93. A. Sawchuk. Space-variant image restoration by coordinate transformation. *J. Opt. Soc. Am.* **6**, 138 (1974).
94. G. Sehaml and D. Rudolph. Holographic diffraction gratings. In: *Progress in Optics*, Vol. 14 (edited by E. Wolf), pp. 195–244. North-Holland, Amsterdam (1977).
95. J. F. Waldup. Space-variant coherent optical processing. *Optical Engng* **19**, 339–346 (1980).
96. K. Yokomori. Dielectric surface relief gratings with high diffraction efficiency. *Appl. Opt.* **23**, 2303–2310 (1984).